Camera Calibration

• Problem: Estimate camera’s extrinsic & intrinsic parameters.
• Method: Use image(s) of known scene.
• Tools:
  – Geometric camera models.
  – SVD and constrained least-squares.
  – Line extraction methods.
Coordinate Frames

Camera Coordinate Frame

Pixel Coordinates

Intrinsic Parameters

Extrinsic Parameters

Image Coordinate Frame

World Coordinate Frame

Why Calibrate?

Calibration: relates points in the image to rays in the scene
Why Calibrate?

Calibration: relates points in the image to rays in the scene

Perspective Camera

\[ \vec{r} = (x, y, z) \]
\[ \vec{r} / f = \vec{r}' / Z \]
\[ \vec{r}' = (X, Y, Z) \]

f: effective focal length:
distance of image plane from O.

\[ x = f * X / Z \]
\[ y = f * Y / Z \]
\[ z = f \]
Extrinsic Parameters

\[ Pc = R(Pw - T) \]

*Translation followed by rotation*

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Extrinsic Parameters (2nd formulation)

\[ Pc = R Pw + T \]

*R same as before  
*T different*

*Rotation followed by translation*
The Rotation Matrix

\[ R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \]

\[ R \cdot R^T = R^T \cdot R = I \Rightarrow \]

Orthonormal Matrix

Degrees of freedom?

\[
I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Intrinsic Parameters

\[
x = \frac{f}{Z} X
\]

\[
y = \frac{f}{Z} Y
\]

The Transformation between Camera and Image Frame Coordinates

Neglecting any geometric distortions possibly introduced by the optics and in the assumption that the CCD array is made of a rectangular grid of photosensitive elements, we have

\[
x = -(e_x - o_x)x_x
\]

\[
y = -(e_y - o_y)y_y
\]

(2.20)

with \((e_x, e_y)\) the coordinates in pixel of the image center (the principal point), and \((o_x, o_y)\) the effective size of the pixel (in millimeters) in the horizontal and vertical direction respectively.

Therefore, the current set of intrinsic parameters is \(f, e_x, e_y, k_x, k_y\).
Image and Camera Frames

Geometric Model

\[ x = \frac{-(x_{im} - o_x) s_x}{f} \]
\[ y = \frac{-(y_{im} - o_y) s_y}{f} \]

- Transformation from Image to Camera Frame.
- Transformation from World to Camera Frame.
- Perspective projection \( (f, R, T) \)
- Point in Camera Frame.

No distortion!
Camera Calibration: Issues

- Which parameters need to be estimated.
  - Focal length, image center, aspect ratio
  - Radial distortions
- What kind of accuracy is needed.
  - Application dependent
- What kind of calibration object is used.
  - One plane, many planes
  - Complicated three dimensional object

Camera Calibration

Calibration object

Extracted features
Camera Calibration

Extract centers of circles

Basic Equations

\[
\begin{align*}
x_{im} &= -\frac{f}{s_x} \frac{X^c}{Z^c} + o_x \\
y_{im} &= -\frac{f}{s_y} \frac{Y^c}{Z^c} + o_y
\end{align*}
\]

\[
\begin{align*}
X^c &= r_{11} X^w + r_{12} Y^w + r_{13} Z^w + T_x \\
Y^c &= r_{21} X^w + r_{22} Y^w + r_{23} Z^w + T_y \\
Z^c &= r_{31} X^w + r_{32} Y^w + r_{33} Z^w + T_z
\end{align*}
\]
Basic Equations

\[
x = -f_x \frac{X^c}{Z^c} + o_x
\]
\[
y = -f_y \frac{Y^c}{Z^c} + o_y
\]

\[
X^c = r_{11} X^w + r_{12} Y^w + r_{13} Z^w + T_x
\]
\[
Y^c = r_{21} X^w + r_{22} Y^w + r_{23} Z^w + T_y
\]
\[
Z^c = r_{31} X^w + r_{32} Y^w + r_{33} Z^w + T_z
\]
Basic Equations

Extrinsic Parameters
1) Rotation matrix R (3x3)
2) Translation vector T (3x1)

Intrinsic Parameters
1) \( fx = \frac{f}{sx} \), length in effective horizontal pixel size units.
2) \( \alpha = \frac{sy}{sx} \), aspect ratio.
3) \((ox, oy)\), image center coordinates.
4) Radial distortion coefficients.

Total number of parameters (excluding distortion): 8

\[ x - o_x = -f_x \frac{r_{11} X^w + r_{12} Y^w + r_{13} Z^w + T_x}{r_{31} X^w + r_{32} Y^w + r_{33} Z^w + T_z} \]
\[ y - o_y = -f_y \frac{r_{21} X^w + r_{22} Y^w + r_{23} Z^w + T_y}{r_{31} X^w + r_{32} Y^w + r_{33} Z^w + T_z} \]

1) Assume that image center is known.
2) Solve for the remaining parameters.
3) Use N image points \((x_i, y_i)\) and their corresponding N world points \([X_i^w, Y_i^w, Z_i^w]^T\).
Basic Equations

\[
x = -f_x \frac{r_{11} X^w + r_{12} Y^w + r_{13} Z^w + T_x}{r_{31} X^w + r_{32} Y^w + r_{33} Z^w + T_z}
\]
\[
y = -f_y \frac{r_{21} X^w + r_{22} Y^w + r_{23} Z^w + T_y}{r_{31} X^w + r_{32} Y^w + r_{33} Z^w + T_z}
\]

(1)

1) Assume that image center is known.
2) Solve for the remaining parameters.
3) Use N image points \((x_i, y_i)\) and their corresponding N world points \([X_i^w, Y_i^w, Z_i^w]^T\)

Basic Equations

\[
x_i f_y (r_{21} X_i^w + r_{22} Y_i^w + r_{23} Z_i^w + T_y)
\]
\[
y_i f_x (r_{11} X_i^w + r_{12} Y_i^w + r_{13} Z_i^w + T_x)
\]

(2)

1) Assume that image center is known.
2) Solve for the remaining parameters.
3) Use N image points \((x_i, y_i)\) and their corresponding N world points \([X_i^w, Y_i^w, Z_i^w]^T\)
Basic Equations

\[ x_i X_i v_1 + x_i Y_i v_2 + x_i Z_i v_3 + x_i v_4 - y_i X_i v_5 - y_i Y_i v_6 - y_i Z_i v_7 - y_i v_8 = 0 \]  \hspace{1cm} (3)

\[
\begin{align*}
    v_1 &= r_{21}, v_5 = \alpha r_{11} \\
    v_2 &= r_{22}, v_6 = \alpha r_{12} \\
    v_3 &= r_{23}, v_7 = \alpha r_{13} \\
    v_4 &= T_y, v_8 = \alpha T_x
\end{align*}
\]

How would we solve this system?
Basic Equations

\[
A = \begin{bmatrix}
    x_1 X^w & x_1 Y^w & x_1 Z^w & x_1 & -y_1 X^w & -y_1 Y^w & -y_1 Z^w & -y_1 \\
    x_2 X^w & x_2 Y^w & x_2 Z^w & x_2 & -y_2 X^w & -y_2 Y^w & -y_2 Z^w & -y_2 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    x_N X^w & x_N Y^w & x_N Z^w & x_N & -y_N X^w & -y_N Y^w & -y_N Z^w & -y_N
\end{bmatrix}
\]

\[
A \mathbf{v} = 0 \quad (3)
\]

How would we solve this system?

Rank of matrix A?

Solution up to a scale factor.

Singular Value Decomposition

Appendix A.6

\[A = U D V^T\]

A: m x n
U: m x m, columns orthogonal unit vectors.
V: n x n , -/-
D: m x n , diagonal. The diagonal elements
\(\sigma_i\) are the singular values
\(\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0\)
Singular Value Decomposition
Appendix A.6

\[ A = UDV^T \]

1. Square A non-singular iff \( \sigma_i \neq 0 \)
2. For square A \( C = \sigma_1/\sigma_N \) is the condition number
3. For rectangular A \# of non-zero \( \sigma_i \) is the rank
4. For square non-singular A:
   \[ A^{-1} = VD^{-1}U^T \]
5. For square A, pseudoinverse:
   \[ A^+ = VD_0^{-1}U^T \]
6. Singular values of A = square roots of eigenvalues of
   \[ AA^T \quad \text{and} \quad A^T A \]
7. Columns of U, V
   Eigenvectors of
   \[ AA^T \quad A^T A \]
8. Frobenius norm of a matrix

If rank(A) = n-1 (7 in our case) then
the solution is the eigenvector which corresponds to the ONLY zero eigenvalue.
Solution up to a scale factor.
Solving for \( v \)

\[
A = \begin{bmatrix}
    x_1 X_v & x_1 Y_v & x_1 Z_v & x_1 & - y_1 X_v & - y_1 Y_v & - y_1 Z_v & - y_1 \\
    x_2 X_v & x_2 Y_v & x_2 Z_v & x_2 & - y_2 X_v & - y_2 Y_v & - y_2 Z_v & - y_2 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    x_N X_v & x_N Y_v & x_N Z_v & x_N & - y_N X_v & - y_N Y_v & - y_N Z_v & - y_N \\
\end{bmatrix}
\]

\[
A \bar{v} = 0 \quad (3)
\]

How would we solve this system: SVD.

**Solution:** \( \bar{v} = \gamma ( r_{21}, r_{22}, r_{23}, T_y, \alpha r_{11}, \alpha r_{12}, \alpha r_{13}, \alpha T_x ) \)

Unknown scale factor \( \gamma = ? \)

Aspect ratio \( \alpha = ? \)

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Solving for \( T_z \) and \( fx \)?
Solving for $T_z$ and $f_x$?

$$x_i (r_{31} X_i^w + r_{32} Y_i^w + r_{33} Z_i^w + T_z) = -f_x (r_{11} X_i^w + r_{12} Y_i^w + r_{13} Z_i^w + T_x)$$

$$A \begin{pmatrix} T_z \\ f_x \end{pmatrix} = b$$

How would we solve this system?

Solving for $T_z$ and $f_x$?

$$x_i (r_{31} X_i^w + r_{32} Y_i^w + r_{33} Z_i^w + T_z) = -f_x (r_{11} X_i^w + r_{12} Y_i^w + r_{13} Z_i^w + T_x)$$

$$A \begin{pmatrix} T_z \\ f_x \end{pmatrix} = b$$

How would we solve this system?

$$\begin{pmatrix} T_z^* \\ f_x^* \end{pmatrix} = (A^T A)^{-1} A^T b$$

Solution in the least squares sense.
Camera Center

Camera Models (linear versions)

\[
M_{int} = \begin{pmatrix}
  -f/s_x & 0 & o_x \\
  0 & -f/s_y & o_y \\
  0 & 0 & 1
\end{pmatrix}
\]

Elegant decomposition.
No distortion!

\[
M_{ext} = \begin{pmatrix}
  r_{11} & r_{12} & r_{13} & -R^T \ T \\
  r_{21} & r_{22} & r_{23} & -R^T \ T \\
  r_{31} & r_{32} & r_{33} & -R^T \ T
\end{pmatrix}
\]

The Linear Matrix Equation of Perspective Projections

Homogeneous Coordinates

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix} = M_{int} \ M_{ext}
\begin{pmatrix}
  X_w \\
  Y_w \\
  Z_w \\
  1
\end{pmatrix}
\]

Measured Pixel \( (x_{im}, y_{im}) \) \(\rightarrow\) World Point \( (X_w, Y_w, Z_w) \)
Camera Calibration – Other method

\[
\begin{bmatrix}
  u \\
  v \\
  w \\
\end{bmatrix} = P 
\begin{bmatrix}
  X \\
  Y \\
  Z \\
  1 \\
\end{bmatrix}
\]

\[
x = \frac{u}{w}
\]

\[
y = \frac{v}{w}
\]

Extracted features

Step 1: Estimate P
Step 2: Decompose P into internal and external parameters R,T,C

Camera Calibration: Step 1

\[
\begin{bmatrix}
  u \\
  v \\
  w \\
\end{bmatrix} = P 
\begin{bmatrix}
  X \\
  Y \\
  Z \\
  1 \\
\end{bmatrix}
\]

\[
x = \frac{u}{w}
\]

\[
y = \frac{v}{w}
\]

\[
wX = u
\]

\[
wY = v
\]

Extracted features

Each point \((x,y)\) gives us two equations

\[
x(p_{31}X + p_{32}Y + p_{33}Z + p_{34}) = p_{11}X + p_{12}Y + p_{13}Z + p_{14}
\]

\[
y(p_{31}X + p_{32}Y + p_{33}Z + p_{34}) = p_{21}X + p_{22}Y + p_{23}Z + p_{24}
\]
Camera Calibration: Step 1

Each corner \((x,y)\) gives us two equations:

\[
x(p_{31}X + p_{32}Y + p_{33}Z + p_{34}) = p_{11}X + p_{12}Y + p_{13}Z + p_{14}
\]

\[
y(p_{31}X + p_{32}Y + p_{33}Z + p_{34}) = p_{21}X + p_{22}Y + p_{23}Z + p_{24}
\]

Extracted features

```
x Y Z 1 0 0 0 0 -xY -xZ -x
0 0 0 0 X Y Z 1 -yX -yY -yZ
\vdots
\vdots
X Y Z 1 0 0 0 0 -xY -xZ -x
0 0 0 0 X Y Z 1 -yX -yY -yZ
```

A

\[
\begin{bmatrix}
    p_{11} & p_{12} & p_{13} & p_{14} \\
    p_{21} & p_{22} & p_{23} & p_{24} \\
    \vdots & \vdots & \vdots & \vdots \\
    p_{n1} & p_{n2} & p_{n3} & p_{n4}
\end{bmatrix}
\]

\[
\begin{bmatrix}
    0 \\
    0 \\
    \vdots \\
    0
\end{bmatrix}
\]

n points gives us 2n equations
Camera Calibration: Step 1

We need to solve
\[ A\mathbf{p} = 0 \]

In the presence of noise we need to solve
\[ \min_{\mathbf{p}} \| A\mathbf{p} \| \]

The solution is given by the eigenvector with the smallest eigenvalue of \( A^T A \)

The result can be improved through non-linear minimization.

\[ \min_{\mathbf{p}} \sum_{i} \left( \left( \frac{x_i - \frac{u_i}{w_i}}{w_i} \right)^2 + \left( \frac{y_i - \frac{v_i}{w_i}}{w_i} \right)^2 \right) \]
Camera Calibration: Step 1

The result can be improved through non-linear minimization.

\[ \min_p \sum_i \left( \left( x_i - \frac{u_i}{w_i} \right)^2 + \left( y_i - \frac{v_i}{w_i} \right)^2 \right) \]

\[
\begin{bmatrix}
  u_i \\
  v_i \\
  w_i
\end{bmatrix}
= P
\begin{bmatrix}
  X_i \\
  Y_i \\
  Z_i
\end{bmatrix}
\]

Minimize the distance between the predicted and detected features.