

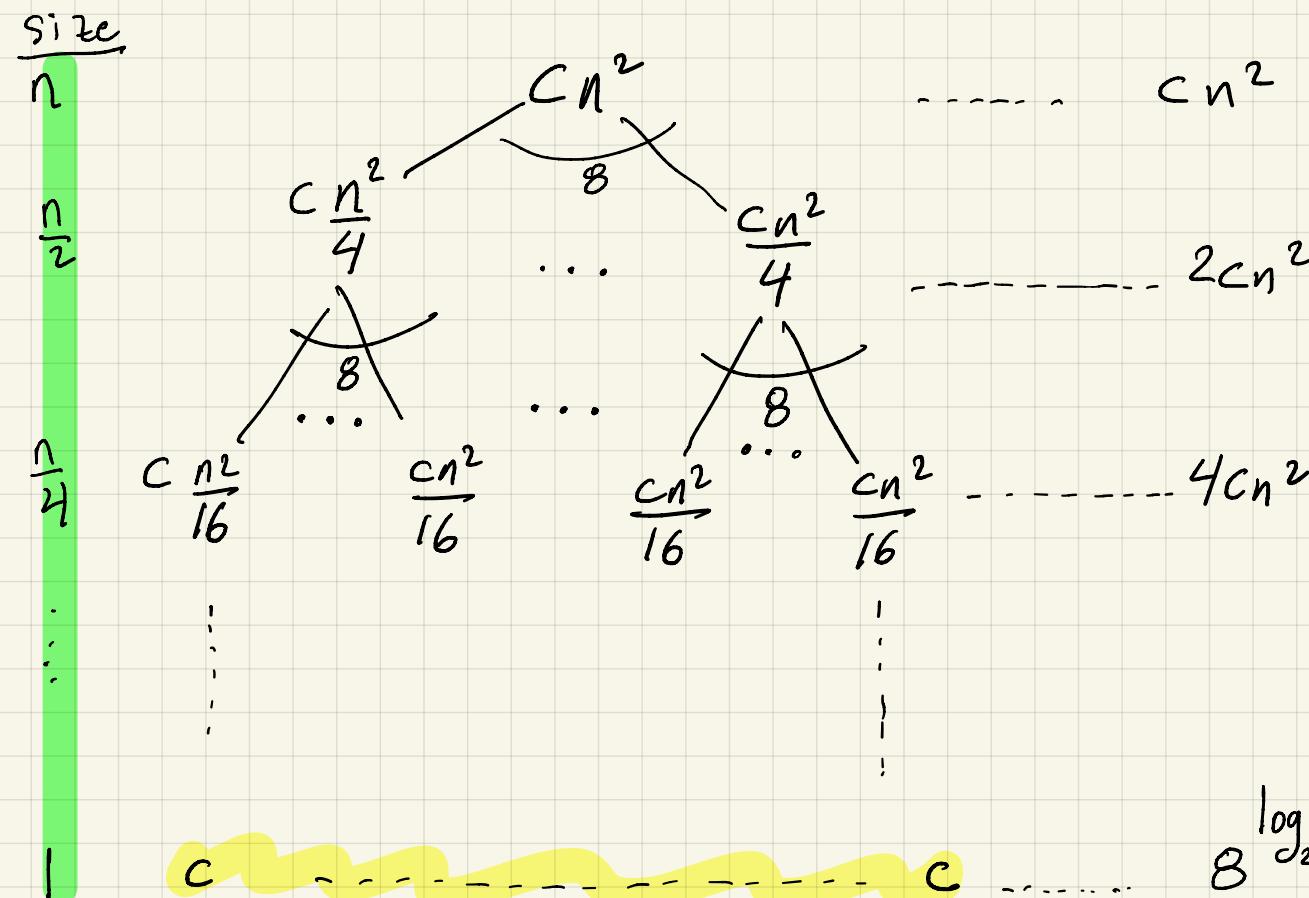
Divide & Conquer for $n \times n$ matrix

$$\begin{array}{c}
 \text{Diagram showing matrix division:} \\
 \begin{bmatrix} R & S \\ T & U \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} E & F \\ G & H \end{bmatrix}
 \end{array}$$

The diagram illustrates the divide-and-conquer algorithm for matrix multiplication. It shows a large $n \times n$ matrix being partitioned into four quadrants: R (top-left), S (top-right), T (bottom-left), and U (bottom-right). Red arrows indicate the width and height of each quadrant. The matrix is multiplied by two smaller matrices, A and B (top row) and C and D (bottom row) respectively, and E and F (left column) and G and H (right column) respectively.

$$\begin{aligned}
 R &= \underline{A \cdot E} + \underline{B \cdot G} \\
 S &= \underline{A \cdot F} + \underline{B \cdot H} \\
 T &= \\
 U &=
 \end{aligned}$$

$$T(n) = \begin{cases} 8T\left(\frac{n}{2}\right) + cn^2 & n > 1 \\ c & n = 1 \end{cases}$$



$$\frac{\# \text{ nodes}}{1} = 8^0$$

$$8 = 8^1$$

$$64 = 8^2$$

$$8^{\log_2 n} C = Cn^{\log_2 8} = Cn^3$$

$$\sum_{i=0}^{\log_2 n - 1} C 8^i \left(\frac{n}{2^i}\right)^2 + C 8^{\log_2 n}$$

$\left(\frac{n}{2^i}\right)^2$

$$= \sum_{i=0}^{\log_2 n - 1} C 2^i n^2 + Cn^3$$

$$= Cn^2 \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + 1 \right) + Cn^3$$

$$= Cn^3 \left(1 + \frac{1}{2} + \frac{1}{4} + \dots \right) = \Theta(n^3)$$

Strassen's alg improves on this by making

$$T(n) = \begin{cases} 7T(n/2) + cn^2 & n > 1 \\ c & n = 1 \end{cases}$$

$$\sum_{i=0}^{\log_2 n - 1} c \left(\frac{7}{4}\right)^i n^2 + c 7^{\log_2 n}$$

Same as before largest term is $n^{\log_2 7} = n^{2.81}$

$$cn^2 \left(\frac{n^{\log_2 7}}{n^2} \cdot \frac{4}{7} + \frac{n^{\log_2 7}}{n^2} \left(\frac{4}{7} \right)^2 + \dots \right) + cn^{\log_2 7}$$

$$= cn^{\log_2 7} \left[1 + \frac{4}{7} + \left(\frac{4}{7} \right)^2 + \dots \right]$$

Strassen's Alg.

$$\begin{bmatrix} R & S \\ T & U \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} E & G \\ F & H \end{bmatrix}$$

$$P_1 = A \cdot (G - H)$$

$$P_2 = (A + B) \cdot H$$

$$P_3 = (C + D) \cdot E$$

$$P_4 = D \cdot (F - E)$$

$$P_5 = (A + D) \cdot (E + F)$$

$$P_6 = (B - D) \cdot (F + H)$$

$$P_7 = (A - C) \cdot (E + G)$$

$$S = P_1 + P_2$$

$$R =$$

$$T =$$

$$U =$$

using
Addition
only

$$A(G - H) + (A + B)H$$

$$\begin{aligned} &= AG - AH + AH + BH \\ &= AG + BH \end{aligned}$$

Growth of Functions

So far: $1.5n^2 + 3.5n - 2 = \Theta(n^2)$

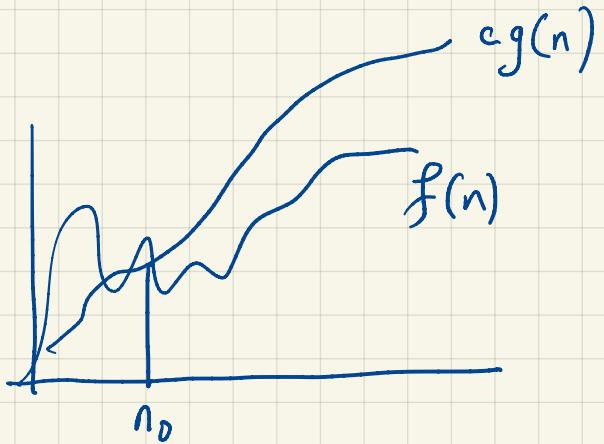
$$n \log n + n = \Theta(n \log n)$$

Asymptotic efficiency: what happens when n is very large.

- Ignore low-order terms
- drop constant factors

O-notation

$\mathcal{O}(g(n)) = \{ f(n) : \exists \text{ positive constants } c \text{ and } n_0 \text{ such that}$
 $0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0 \}$



$g(n)$ is an asymptotic upper bound on $f(n)$

When we write $f(n) = \mathcal{O}(g(n))$ what we mean is that

$$f(n) \in \mathcal{O}(g(n))$$

Example: $2n^2 = O(n^3)$ $c=1$, $n_0=2$

$$2n^2 \leq 1 \cdot n^3 \text{ for all } n \geq 2$$

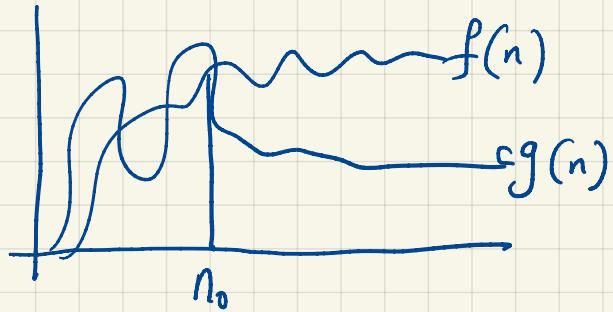
Example function in $O(n^2)$.

$$n^2 \quad n^2 + n \quad n^2 + 1000n \quad n^{1.9}$$

$$\frac{n^2}{\log n}$$

Ω -notation

$\Omega(g(n)) = \{ f(n) : \exists \text{ positive constants } c \text{ and } n_0 \text{ such that}$
 $0 \leq cg(n) \leq f(n) \text{ for all } n \geq n_0 \}$



$g(n)$ is an asymptotic lower bound
on $f(n)$

Example: $\sqrt{n} = \Omega(\lg n)$ $c=1$ $n_0 = 16$

$$\sqrt{n} \geq 1 \cdot \log_2 n \text{ for all } n \geq 16$$

Example functions in $\mathcal{Q}(n^2)$

$$n^2$$

$$n^2+n$$

$$n^2-n$$

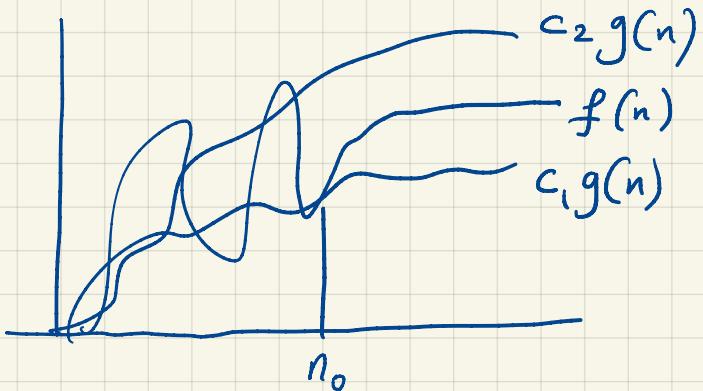
$$\left(n^2-n \geq \frac{1}{2}n^2 \text{ for large } n \right)$$

$$n^2 - 100n$$

$$n^2 \log n$$

Θ -notation

$\Theta(g(n)) = \{f(n) : \exists \text{ positive constants } c_1, c_2 \text{ and } n_0 \text{ such that}$
 $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \text{ for all } n \geq n_0\}$



$g(n)$ is asymptotic tight bound
on $f(n)$

Example: $\frac{n^2}{2} - 2n = \Theta(n^2)$

$$\boxed{\frac{1}{4}} n^2 \leq \frac{n^2}{2} - 2n \leq \boxed{\frac{1}{2}} n^2$$

$n_0 = 8$

Transitivity: $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$

$$\text{then } f(n) = \Theta(h(n))$$

this is true for O and Ω

Reflexivity: $f(n) = \Theta(f(n))$, same for O and Ω

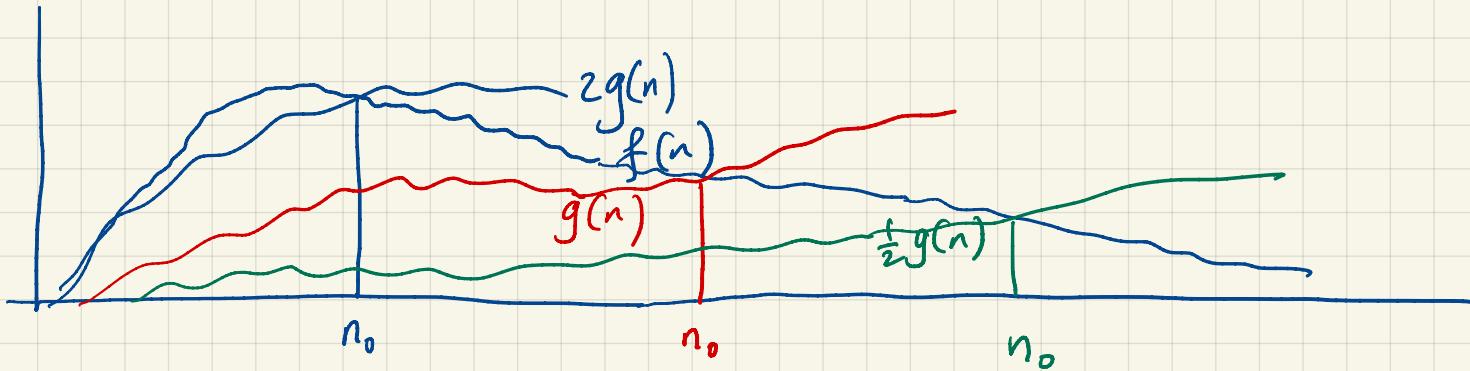
Symmetry: $f(n) = \Theta(g(n)) \Leftrightarrow g(n) = \Theta(f(n))$

$$f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n))$$

$$f(n) = \Theta(g(n)) \Leftrightarrow \begin{cases} f(n) = O(g(n)) \\ f(n) = \Omega(g(n)) \end{cases}$$

O-notation

$\mathcal{O}(g(n)) = \{f(n): \text{for all constants } c > 0, \exists \text{ a constant } n_0 > 0 \text{ such that } 0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0\}$



$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0 \text{ (zero)}$$

$$n^{1.99} = \mathcal{O}(n^2)$$

$$\frac{n^2}{\log n} = \mathcal{O}(n^2)$$

$$n^2 \neq \mathcal{O}(n^2)$$

$$n^2 = \mathcal{O}(n^2)$$

ω -notation: (symmetric)

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty$$

Abuse of notation

$$f(n) = O(g(n))$$

$$f(n) \leq g(n)$$

$$f(n) = \Omega(g(n))$$

$$f(n) \geq g(n)$$

$$f(n) = \Theta(g(n))$$

$$f(n) = g(n)$$

$$f(n) = o(g(n))$$

$$f(n) < g(n)$$

$$f(n) = \omega(g(n))$$

$$f(n) > g(n)$$

Merge Sort
 $n \log n$

$$n \log n = o(n^2)$$

$$n \log n = O(n^2)$$

Insertion
 n^2

Selection Sort
 n^2

$$n^2 \neq o(n^2)$$

$$n^2 = \Theta(n^2) \quad n^2 = O(n^2) \quad n^2 = \Omega(n^2)$$

Two important facts: $\underbrace{n^b}_{\text{poly.}} = o(\underbrace{a^n}_{\text{exponential}}) \quad a > 1$

$$\log^b n = o(n^a) \quad a > 0$$

Some useful information about logarithms

- $\log_b a = \frac{\log_c a}{\log_c b}$ Constant

so $\log_b a = \Theta(\log a)$

- $n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n (1 + \Theta(\frac{1}{n}))$ [Stirling Approx.]

so $\log n! = \Theta(n \log n)$

- $a^{\log b} = b^{\log a}$

so $7^{\log_2 n} = n^{\log_2 7} = n^{2.81\dots} = o(n^3)$