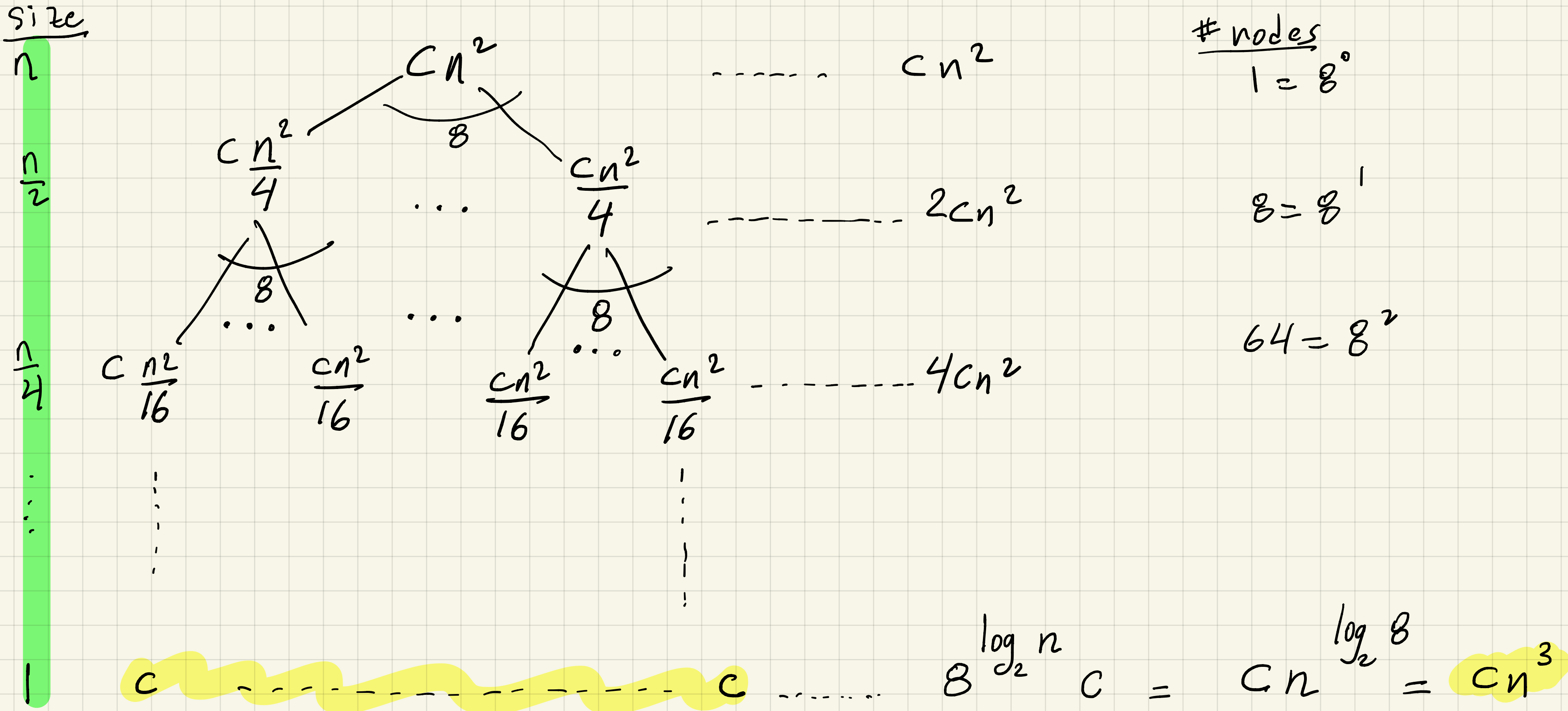
Divide & Conquer for $n \times n$ matrix

$$\begin{array}{cc} & \begin{array}{cc} \overset{\longleftrightarrow}{\frac{n}{2}} & \end{array} \\ n & \left[\begin{array}{c|c} R & S \\ \hline T & U \end{array}\right] \end{array} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \times \left[\begin{array}{c|c} E & F \\ \hline G & H \end{array}\right]$$

$$R = \underline{A.E} + \underline{B.G}$$
$$S = \underline{A.F} + \underline{B.H}$$
$$T =$$
$$U =$$

$$T(n) = \begin{cases} 8T(n/2) + cn^2 & n > 1 \\ c & n = 1 \end{cases}$$



$$\sum_{i=0}^{\log_2 n - 1} c\, 8^i \left(\frac{n}{2^i}\right)^2 + c\, 8^{\log_2 n} = \sum_{i=0}^{\log_2 n - 1} c\, 2^i n^2 + cn^3$$

$$= cn^2 \left(\underbrace{\frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \cdots + 1}_{} \right) + cn^3$$

$$= cn^3 \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots \right) = \Theta(n^3)$$

Strassen' alg improves on this by making

$$T(n) = \begin{cases} 7T(n/2) + cn^2 & n > 1 \\ c & n = 1 \end{cases}$$

$$\sum_{i=0}^{\log_2 n - 1} c\left(\frac{7}{4}\right)^i n^2 + c\,7^{\log_2 n}$$

same as before largest term is $n^{\log_2 7} = n^{2.81}$

$$cn^2\left(\frac{n^{\log_2 7}}{n^2}\cdot\frac{4}{7} + \frac{n^{\log_2 7}}{n^2}\left(\frac{4}{7}\right)^2 + \cdots\right) + cn^{\log_2 7}$$

$$= cn^{\log_2 7}\left[1 + \tfrac{4}{7} + \left(\tfrac{4}{7}\right)^2 + \cdots\right]$$

Strassen's Alg.

$$\begin{bmatrix} R & S \\ T & U \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} E & G \\ F & H \end{bmatrix}$$

$$P_1 = A \cdot (G - H)$$

$$P_2 = (A + B) \cdot H$$

$$P_3 = (C + D) \cdot E$$

$$P_4 = D \cdot (F - E)$$

$$P_5 = (A + D) \cdot (E + F)$$

$$P_6 = (B - D) \cdot (F + H)$$

$$P_7 = (A - C) \cdot (E + G)$$

$S = P_1 + P_2$

$R =$

$T =$

$U =$

using Addition only

$$A(G - H) + (A + B) H$$
$$= AG - \cancel{AH} + \cancel{AH} + BH$$
$$= AG + BH$$

Growth of Functions

So far: $1.5n^2 + 3.5n - 2 = \Theta(n^2)$

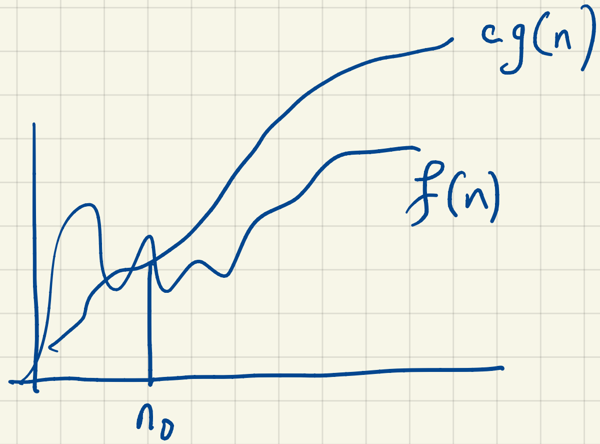
$n\log n + n = \Theta(n\log n)$

Asymptotic efficiency: what happens when n is very large.

- Ignore low-order terms
- drop constant factors

O-notation

$$O(g(n)) = \{ f(n) : \exists \text{ positive constants } c \text{ and } n_0 \text{ such that}$$
$$0 \leq f(n) \leq c g(n) \text{ for all } n \geq n_0 \}$$



$g(n)$ is an asymptotic upper bound on $f(n)$

When we write $f(n) = O(g(n))$ what we mean is that

$$f(n) \in O(g(n))$$

Example: $2n^2 = O(n^3)$ $\quad c=1, \; n_0 = 2$

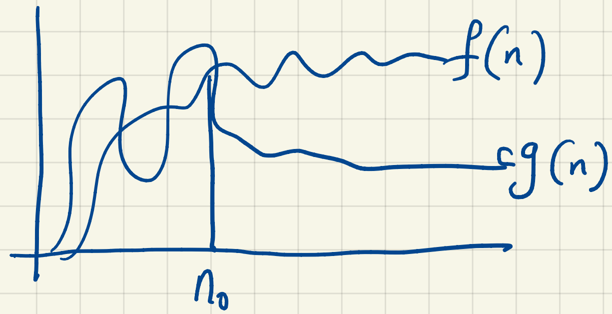
$$2n^2 \leq 1 \cdot n^3 \text{ for all } n \geq 2$$

Example function in $O(n^2)$.

$n^2 \qquad n^2 + n \qquad n^2 + 1000n \qquad n^{1.9} \qquad \dfrac{n^2}{\log n}$

Ω-notation

$$\Omega(g(n)) = \{ f(n) : \exists \text{ positive constants } c \text{ and } n_0 \text{ such that } 0 \leq cg(n) \leq f(n) \text{ for all } n \geq n_0 \}$$



$g(n)$ is an asymptotic lower bound on $f(n)$

Example: $\sqrt{n} = \Omega(\lg n)$ $\quad c = 1 \quad n_0 = 16$

$$\sqrt{n} \geq 1 \cdot \log_2 n \quad \text{for all } n \geq 16$$

Example functions in $\Omega(n^2)$

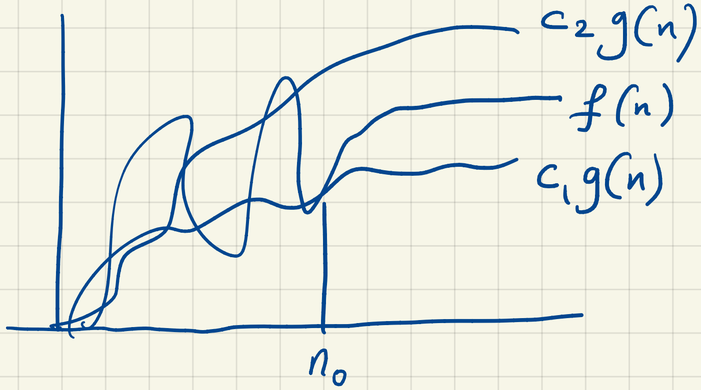
n^2 $\qquad$ n^2+n $\qquad$ n^2-n $\qquad$ ($n^2-n \geq \frac{1}{2}n^2$ for large n)

n^2-100n $\qquad$ $n^2\log n$

Θ - notation

$\Theta(g(n)) = \{ f(n) : \exists$ positive constants c_1, c_2 and n_0 such that

$0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0 \}$



$g(n)$ is asymptotic $\underline{\text{tight}}$ bound on $f(n)$

Example: $\frac{n^2}{2} - 2n = \Theta(n^2)$

$$\boxed{1/4}\, n^2 \le \frac{n^2}{2} - 2n \le \boxed{1/2}\, n^2$$

$\underline{n_0 = 8}$

Transivity: $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$

then $f(n) = \Theta(h(n))$

this is true for O and Ω

Reflexivity: $f(n) = \Theta(f(n))$, same for O and Ω

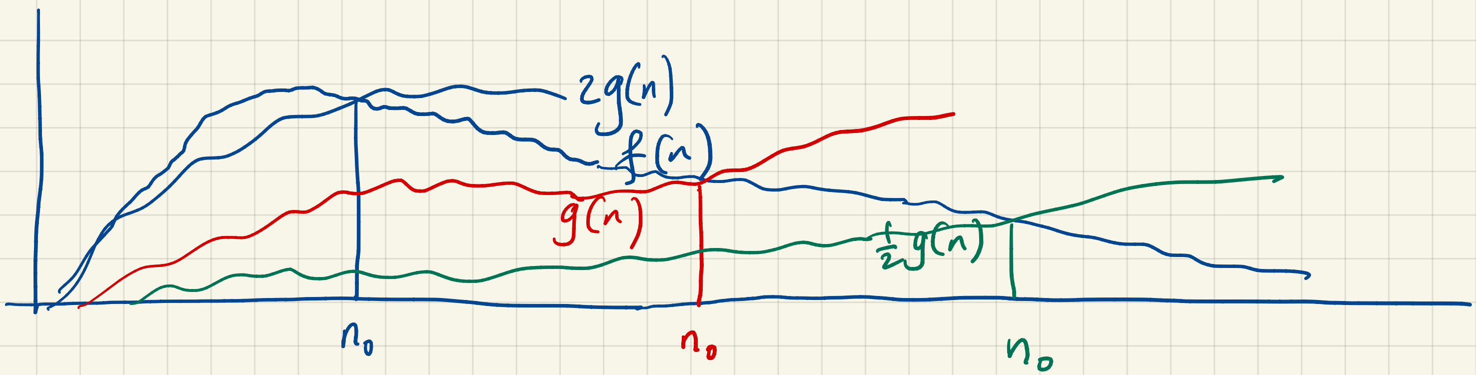
Symmetry: $f(n) = \Theta(g(n)) \Longleftrightarrow g(n) = \Theta(f(n))$

$f(n) = O(g(n)) \Longleftrightarrow g(n) = \Omega(f(n))$

$$f(n) = \Theta(g(n)) \Longleftrightarrow \begin{cases} f(n) = O(g(n)) \\ f(n) = \Omega(g(n)) \end{cases}$$

o-notation

$$o(g(n)) = \{ f(n) : \text{for all constants } c > 0, \exists \text{ a constant } n_0 > 0 \text{ such that } 0 \leq f(n) < cg(n) \text{ for all } n \geq n_0 \}$$



$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \quad \text{(zero)}$$

$$n^{1.99} = o(n^2) \qquad \frac{n^2}{\log n} = o(n^2)$$

$$n^2 \neq o(n^2) \qquad n^2 = O(n^2)$$

ω-notation: (symmetric) $\displaystyle \lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$

Abuse of notation

$f(n) = O(g(n))$	$f(n) \leq g(n)$
$f(n) = \Omega(g(n))$	$f(n) \geq g(n)$
$f(n) = \Theta(g(n))$	$f(n) = g(n)$
$f(n) = o(g(n))$	$f(n) < g(n)$
$f(n) = \omega(g(n))$	$f(n) > g(n)$

Merge Sort	Insertion	Selection Sort
$n \log n$	n^2	n^2

$n \log n = o(n^2)$ $\qquad$ $n^2 \neq o(n^2)$

$n \log n = O(n^2)$ $\qquad$ $n^2 = \Theta(n^2)$ $\qquad$ $n^2 = O(n^2)$ $\qquad$ $n^2 = \Omega(n^2)$

Two important facts: $\underbrace{n^b}_{\text{poly.}} = o(\underbrace{a^n}_{\text{exponential}}) \qquad a > 1$

$\text{logarithmic} \rightarrow \log^b n = o(n^a) \qquad a > 0$

Some useful information about logarithms

- $$\log_b a = \frac{\log_c a}{\log_c b}$$ Constant

  so $\log_b a = \Theta(\log a)$

- $$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\Theta\left(\frac{1}{n}\right)\right)$$ [Stirling Approx.]

  so $\log n! = \Theta(n \log n)$

- $$a^{\log b} = b^{\log a}$$

  So $7^{\log_2 n} = n^{\log_2 7} = n^{2.81\ldots} = o(n^3)$