Mixture of priors

Summary: Given $f(\mu) = \alpha g(\mu) + \beta h(\mu)$ [mixed prior]

Find: $-$ $g(\mu|x)$ [posterior due to $g(\mu)$]

$-$ $h(\mu|x)$ [posterior due to $h(\mu)$]

Then $f(\mu|x) = \alpha(x)\, g(\mu|x) + \beta(x)\, h(\mu|x)$

where $\alpha(x)$ prop. to $\alpha \int f(x|\mu) g(\mu)\, d\mu = \alpha g(x)$

$\beta(x)$ prop. to $\beta \int f(x|\mu) h(\mu)\, d\mu = \beta h(x)$

$\alpha(x) + \beta(x) = 1$

Example: Assume $x|\mu \sim N(\mu, \sigma^2)$

and $g(\mu) : N(\dot\beta, \tau^2)$

$h(\mu) \propto 1$ (improper prior)

We know:

$$g(\mu|x) : N\left(\frac{\sigma^2\dot\beta + \tau^2 x}{\sigma^2 + \tau^2}, \frac{\sigma^2\tau^2}{\sigma^2+\tau^2}\right)$$

$$h(\mu|x) : N(x, \sigma^2) \qquad [\text{as if } \tau^2 \longrightarrow \infty]$$

$$\beta(x) \propto \beta \int_{-\infty}^{+\infty} f(x|\mu) \cdot 1 \cdot d\mu = \beta$$

$$\alpha(x) \propto \alpha \int_{-\infty}^{+\infty} f(x|\mu)\, g(\mu)\, d\mu = \alpha \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi}\tau} e^{-\frac{(\mu-\dot\beta)^2}{2\tau^2}} d\mu$$

Rearrange to make form $\; e^{-\frac{[\mu - \;]^2}{2\;^2}}$

Consider $$\frac{(x-\mu)^2}{2\sigma^2} + \frac{(\mu-\dot{\beta})^2}{2\tau^2}$$

$$= \frac{x^2+\mu^2-2\mu x}{2\sigma^2} + \frac{\mu^2+\beta^2-2\mu\dot{\beta}}{2\tau^2}$$

$$= \frac{\tau^2x^2+\tau^2\mu^2-2\mu\tau^2x+\sigma^2\mu^2+\sigma^2\dot{\beta}^2-2\mu\sigma^2\dot{\beta}}{2\sigma^2\tau^2}$$

$$= \frac{\mu^2\left[\sigma^2+\tau^2\right]-2\mu\left[\tau^2x+\sigma^2\dot{\beta}\right]+\tau^2x^2+\sigma^2\dot{\beta}^2}{2\sigma^2\tau^2}$$

$$= \frac{\left[\mu-\frac{\tau^2x+\sigma^2\beta}{\sigma^2+\tau^2}\right]^2}{2\,\frac{\sigma^2\tau^2}{\sigma^2+\tau^2}} - \frac{\left(\frac{\tau^2x+\sigma^2\dot{\beta}}{\sigma^2+\tau^2}\right)^2}{\frac{2\sigma^2\tau^2}{\sigma^2+\tau^2}} + \frac{\tau^2x^2+\sigma^2\dot{\beta}^2}{2\sigma^2\tau^2}$$

$$= \frac{\left[\mu-\quad\right]^2}{2\sigma^2\tau^2/\sigma^2+\tau^2} + \frac{(x-\dot{\beta})^2}{2(\sigma^2+\tau^2)}$$

$$\alpha(x) \propto \alpha \int \frac{\sqrt{\sigma^2+\tau^2}}{\sqrt{2\pi}\,\sigma\tau\sqrt{\sigma^2+\tau^2}}\, e^{-\frac{\left[\mu - \quad\right]^2}{2\frac{\sigma^2\tau^2}{\sigma^2+\tau^2}}}\, \frac{1}{\sqrt{2\pi}}\, e^{-\frac{(x-\dot{\beta})^2}{2(\sigma^2+\tau^2)}}\, d\mu$$

$$= \underbrace{\alpha \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2+\tau^2}}\, e^{-\frac{(x-\dot{\beta})^2}{2(\sigma^2+\tau^2)}}}_{N(\dot{\beta},\, \sigma^2+\tau^2)}$$

$[f(x)$ due to $g(\mu)]$

$$\alpha(x) = \frac{\alpha N(\dot{\beta}, \sigma^2+\tau^2)}{\alpha N(\dot{\beta}, \sigma^2+\tau^2) + \beta} \qquad \beta(x) = \frac{\beta}{\alpha N(\dot{\beta}, \sigma^2+\tau^2) + \beta}$$

We learned something:

$$f(x|\mu) \sim N(\mu, \sigma^2)$$

$$\mu \sim N(\beta, \tau^2)$$

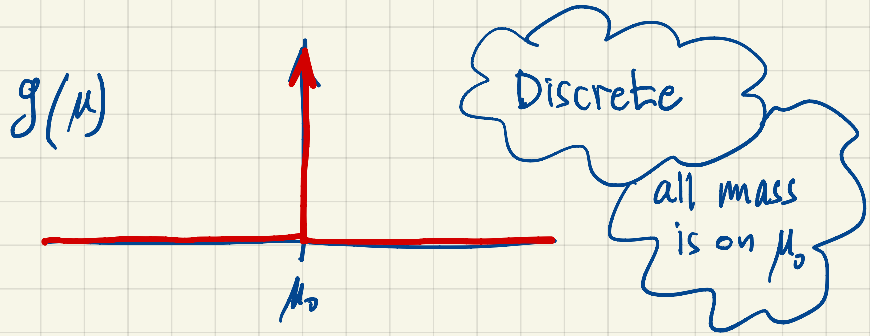
$$f(x) = \int_{-\infty}^{+\infty} f(x|\mu) f(\mu)\, d\mu = \underbrace{\frac{1}{\sqrt{2\pi}\sqrt{\sigma^2+\tau^2}} e^{-\frac{(x-\beta)^2}{2(\sigma^2+\tau^2)}}}_{N(\beta,\ \sigma^2+\tau^2)}$$

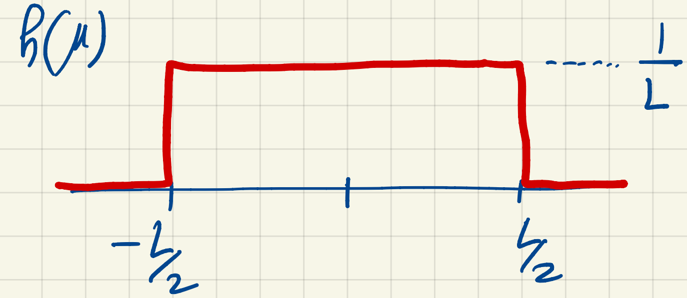
Mixture of Discrete & Continuous

Example:

$$\mu = \begin{cases} \mu_0 & p \\ \text{Unif}\left(-\frac{L}{2}, \frac{L}{2}\right) & 1-p \end{cases}$$

$$\bar{x} \mid \mu \sim N\left(\mu, \sigma^2/n\right)$$



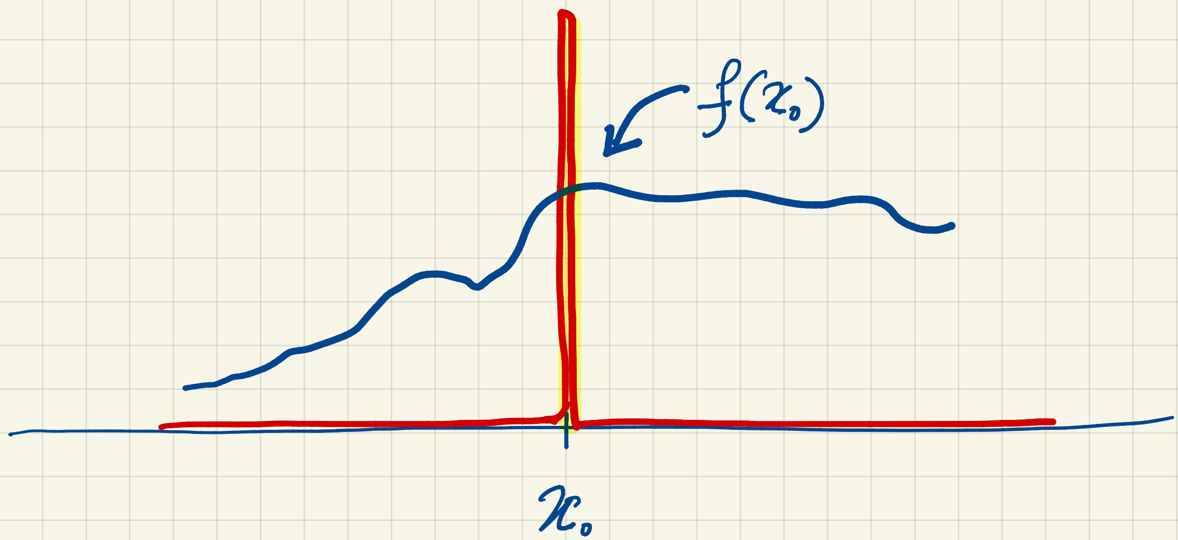


$$g(\mu) = \delta(\mu - \mu_0) \quad \text{where} \quad \delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

Handle discrete with density: $\displaystyle\int_{-\infty}^{+\infty} \delta(x - x_0)\, dx = 1$.

Replace x by x_0 in $f(x)$

More generally: $$\int_R \delta(x - x_0) f(x)\, dx = \begin{cases} f(x_0) & x_0 \in R \\ 0 & \text{otherwise} \end{cases}$$



Area under $\delta(x-x_0)\,f(x) = 1 \cdot f(x_0) = f(x_0)$

Is it Consistent?

$$f(\mu) = p\,\delta(\mu-\mu_0) + (1-p)\frac{1}{L} \qquad -\frac{L}{2} \leq \mu \leq \frac{L}{2}$$

$$P(\mu=\mu_0) = \int_{\mu_0}^{\mu_0} f(\mu)\,d\mu = \int_{\mu_0}^{\mu_0} p\,\delta(\mu-\mu_0)\,d\mu + \int_{\mu_0}^{\mu_0} p\,\frac{1}{L}\,d\mu$$

$$= \quad p \quad + \quad 0 \quad = p \quad \checkmark$$

Example: Coin with $p(H) = p$

$$f(x) = p\,\delta(x-1) + (1-p)\,\delta(x)$$

$$P(x=0) = \int_0^0 f(x)\,dx = \int_0^0 p\,\delta(x-1)\,dx + \int_0^0 (1-p)\,\delta(x)\,dx = 0 + 1-p.$$

$$P(x=1) = \int_1^1 f(x)\,dx = p$$

$$P(x=a) = \int_a^a f(x)\,dx = \int_a^a \left[p\,\delta(x-1) + (1-p)\,\delta(x)\right]dx = 0 \text{ if } 1 \notin [a,a] \text{ and } 0 \notin [a,a]$$

What is posterior due to $g(\mu)$? Same ! [Nothing can change this prior]

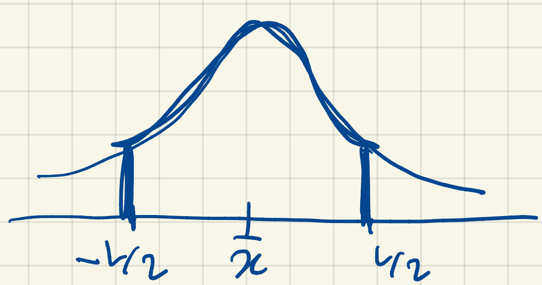
$$g(\mu|\bar{x}) = \frac{f(\bar{x}|\mu)\, g(\mu)}{\int_{-\infty}^{+\infty} f(\bar{x}|\mu)\, g(\mu)\, d\mu} = \frac{f(\bar{x}|\mu)\, \delta(\mu-\mu_0)}{\int_{-\infty}^{+\infty} f(\bar{x}|\mu)\, \delta(\mu-\mu_0)\, d\mu}$$

$$= \frac{f(\bar{x}|\mu)\, \delta(\mu-\mu_0)}{f(\bar{x}|\mu_0)} = \begin{cases} 1.\delta(0) & \mu=\mu_0 \\ 0 & \text{otherwise} \end{cases} = \delta(\mu-\mu_0)$$

What is posterior due to $h(\mu)$?

$$h(\mu|\bar{x}) = \frac{f(\bar{x}|\mu)\, 1/L}{\int_{-L/2}^{L/2} f(\bar{x}|\mu)\, 1/L\, d\mu} \qquad \left(\text{scaled Normal in } \left[-\tfrac{L}{2}, \tfrac{L}{2}\right]\right)$$

Denominator is : $\Phi\left(\dfrac{L/2 - \bar{x}}{\sigma}\right) - \Phi\left(\dfrac{-L/2 - \bar{x}}{\sigma}\right)$



Lindley's Paradox: (Say that μ_0 is most likely no matter what!)

$$\frac{\alpha(\bar{x})}{\beta(\bar{x})} = \frac{\alpha(\bar{x})}{1-\alpha(\bar{x})} = \frac{p\int_{-\infty}^{+\infty} f(\bar{x}|\mu)\,\delta(\mu-\mu_0)\,d\mu}{(1-p)\int_{-L/2}^{L/2} f(\bar{x}|\mu)\,\frac{1}{L}\,d\mu}$$

$$\geqslant \frac{p\int_{-\infty}^{+\infty} f(\bar{x}|\mu)\,\delta(\mu-\mu_0)\,d\mu}{(1-p)\int_{-\infty}^{+\infty} f(\bar{x}|\mu)\,\frac{1}{L}\,d\mu} = \frac{Lp}{1-p} f(\bar{x}|\mu_0)$$

Let $\bar{x} = \mu_0 + c\sigma/\sqrt{n}$. Then $f(\bar{x}|\mu_0) = \frac{\sqrt{n}}{\sqrt{2\pi}\,\sigma} e^{-c^2/2}$

$$\frac{\alpha(\bar{x})}{1-\alpha(\bar{x})} = \frac{Lp}{1-p}\,\frac{\sqrt{n}}{\sqrt{2\pi}\,\sigma}\,e^{-c^2/2} \longrightarrow \infty \text{ when } n \longrightarrow \infty$$

$\lim_{n\to\infty} \alpha(\mu_0 + c\sigma/\sqrt{n}) = 1$, So posterior $\approx \delta(\mu-\mu_0)$

Paradox: We observe $\bar{x}$ c s.d away from it's mean μ_0, but $P(\mu=\mu_0|\bar{x}) = 1$

Another approach: Handle prob. & density in Bayes.

$$P(\mu=\mu_0 \mid \bar{x}) = \frac{f(\bar{x} \mid \mu=\mu_0)\, P(\mu=\mu_0)}{f(\bar{x} \mid \mu=\mu_0)\, P(\mu=\mu_0) + f(\bar{x} \mid \mu\neq\mu_0)\, P(\mu\neq\mu_0)}$$

Now: $f(\bar{x} \mid \mu\neq\mu_0) = \int_{-L/2}^{L/2} f(\bar{x} \mid \mu)\, f(\mu)\, d\mu$

$$P(\mu=\mu_0 \mid \bar{x}) = \frac{p\, f(\bar{x} \mid \mu=\mu_0)}{p\, f(\bar{x} \mid \mu=\mu_0) + (1-p)\int_{-L/2}^{L/2} f(\bar{x} \mid \mu)\, \frac{1}{L}\, d\mu}$$

$$\geqslant \frac{p\, f(\bar{x} \mid \mu=\mu_0)}{p\, f(\bar{x} \mid \mu=\mu_0) + \frac{1-p}{L}}$$

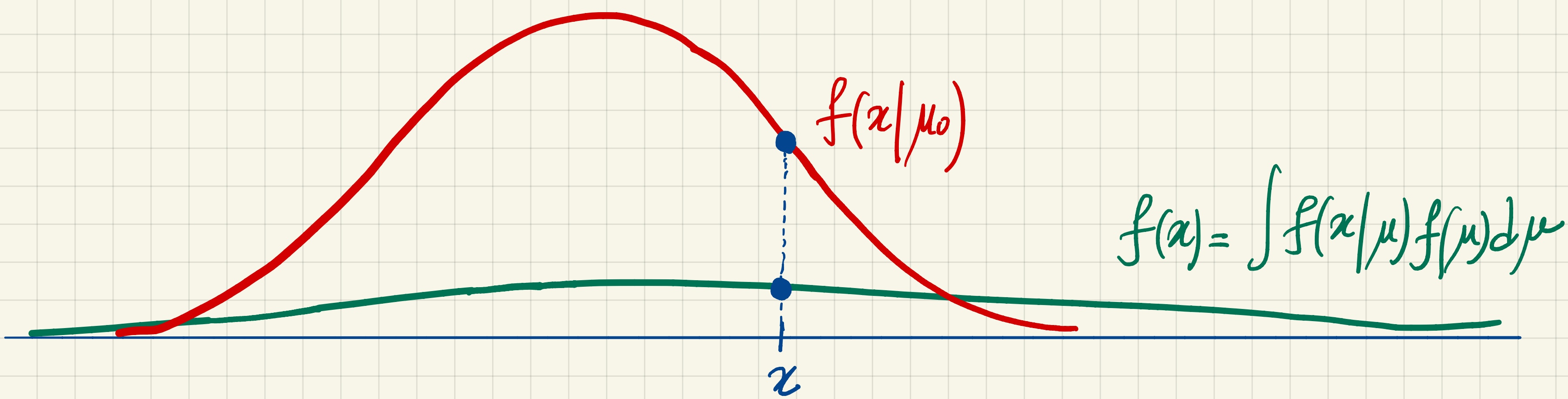
$$\bar{x} = \mu_0 + c\sigma/\sqrt{n} \Rightarrow P(\mu=\mu_0 \mid \bar{x}) \geqslant \frac{p \frac{\sqrt{n}}{\sqrt{2\pi}\,\sigma} e^{-c^2/2}}{p \frac{\sqrt{n}}{\sqrt{2\pi}\,\sigma} e^{-c^2/2} + \frac{1-p}{L}} \to 1$$

Interpretation of Paradox

x is observed with variance σ^2

μ has distribution with variance τ^2

$\sigma^2 \ll \tau^2$ (in our example $\sigma^2/n \to 0$)



Ratio: $\dfrac{f(x|\mu_0)}{f(x) = \int f(x|\mu) f(\mu) d\mu}$ becomes large.