The t-distribution (student)

- $Z \sim N(0,1)$
- $V \sim \chi^2_k$
- Z and V are independent

Then $t = \dfrac{Z}{\sqrt{V/k}}$ satisfies $f(t) \propto \left(1 + \dfrac{t^2}{k}\right)^{-\frac{k+1}{2}}$ [look up constant]

Observation: when k is large, this is almost Normal.

$$\lim_{k \to \infty} \left(1 + \underbrace{\frac{k+1}{k}}_{\downarrow \, 1} \cdot \frac{t^2}{k+1}\right)^{-\frac{k+1}{2}} = e^{-t^2/2}$$

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n \to e^x$$

$E[t] = 0$ if $k > 1$

$$\mathrm{Var}[t] = \begin{cases} \dfrac{k}{k-2} & k > 2 \\ \infty & 1 < k \leq 2 \end{cases}$$

Fisher proved the following for Normal samples

- $(n-1)s^2/\sigma^2 \sim \chi^2_{n-1}$ $\qquad \dfrac{S}{\sigma^2} \sim \chi^2_{n-1}$ where $S = \sum(x_i - \bar{x})^2$ (*)
- $\bar{x}$ and s^2 are independent (only Normal satisfies this)
- (we know) $\dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

So $$\frac{\bar{x}-\mu}{s/\sqrt{n}} \sim t_{n-1}$$ (t-distributed with degree $n-1$)

Remark: The text in Note 8 contains proofs for the first two facts when $n=2$.

(*) This is different than previous definition $S = \sum(x_i-\mu)^2$ when μ was known.

Two teams revisited (unknown σ^2)

$$\frac{(n-1)s_x^2}{\sigma_x^2} \sim \chi^2_{n-1} \qquad \frac{(m-1)s_y^2}{\sigma_y^2} \sim \chi^2_{m-1}$$

So $$\frac{(n-1)s_x^2}{\sigma_x^2} + \frac{(m-1)s_y^2}{\sigma_y^2} \sim \chi^2_{n+m-2} \quad [\text{sum of ind. } \chi^2]$$

From before $$\frac{\bar{x}-\bar{y}-(\mu_x-\mu_y)}{\sqrt{\sigma_x^2/n + \sigma_y^2/m}} \sim N(0,1)$$

Assuming $\sigma_x^2 = \sigma_y^2$ [Though both are unknown], testing $\mu_x = \mu_y$ we get

$$\frac{\bar{x}-\bar{y}}{\sqrt{\left(\frac{1}{n}+\frac{1}{m}\right)\frac{(n-1)s_x^2+(m-1)s_y^2}{n+m-2}}} \sim t_{n+m-2}$$

[Note 8 has an example]

Conjugate Prior

$$x_i \mid \mu, \sigma^2 \sim N(\mu, \sigma^2) \quad \text{are i.i.d.}$$

$$\frac{S_0}{\sigma^2} \sim \chi^2_{k_0}$$

$$\mu \mid \sigma^2 \sim N\left(\beta, \frac{\sigma^2}{n_0}\right)$$

* Making μ & σ independent does not lead to conjugate

We want to prove posterior has the same form.

$$\frac{S'}{\sigma^2} \,\Big|\, x_1 \dots x_n \sim \chi^2_{k} \qquad (\text{Find } S' \text{ and } k)$$

$$\mu \mid \sigma^2, x_1 \dots x_n \sim N\left(\beta', \frac{\sigma^2}{n'}\right) \qquad (\text{Find } \beta' \text{ and } n').$$

First, let's find $\mu \mid \sigma^2, x_1 \dots x_n$

$$\underbrace{f(\mu \mid x_1 \dots x_n, \sigma^2)}_{?} \propto \underbrace{f(x_1 \dots x_n \mid \mu, \sigma^2)}_{\text{Normal}} \cdot \underbrace{f(\mu \mid \sigma^2)}_{\text{Normal}}$$

This is the same setting where μ is unknow but σ^2 known

$$\mu \mid \sigma^2, x_1 \dots x_n \sim N\left(\frac{\sigma^2 \beta / n + \sigma^2/n_0 \, \bar{x}}{\sigma^2/n + \sigma^2/n_0}, \frac{\sigma^2/n \; \sigma^2/n_0}{\sigma^2/n + \sigma^2/n_0} \right)$$

$$\mu \mid \sigma^2, x_1 \dots x_n \sim N\left(\frac{n_0 \beta + n\bar{x}}{n+n_0}, \frac{\sigma^2}{n+n_0} \right) = N\left(\beta', \frac{\sigma^2}{n'}\right)$$

- β' is weighted average of β and $\bar{x}$
- n = # observations
- n_0 = # "observations" in prior.

What about σ^2? Let's start with

$$f\left(\mu, \frac{1}{\sigma^2}\right) \propto \underbrace{\frac{1}{\sigma} e^{-\frac{n_0(\mu-\beta)^2}{2\sigma^2}}}_{f(\mu|\sigma^2)} \cdot \underbrace{\left(\frac{1}{\sigma^2}\right)^{\frac{k_0}{2}-1} e^{-\frac{S_0}{2\sigma^2}}}_{f(1/\sigma^2)}$$

$$= \left(\frac{1}{\sigma^2}\right)^{\frac{k_0+1}{2}-1} e^{-\frac{n_0(\mu-\beta)^2+S_0}{2\sigma^2}}$$

Observe that $f\left(\frac{1}{\sigma^2} \middle| \mu\right) \propto f\left(\mu, \frac{1}{\sigma^2}\right)$

so $\quad \dfrac{S_0+n_0(\mu-\beta)^2}{\sigma^2} \Big| \mu \sim \chi^2_{k_0+1}$ (Another way of looking at prior for σ^2)

Let's find $\frac{1}{\sigma^2} \mid \mu, x_1 \dots x_n$

$$\underbrace{f\left(\frac{1}{\sigma^2} \,\Big|\, x_1 \dots x_n, \mu\right)}_{?} \propto \underbrace{f(x_1 \dots x_n \mid \sigma^2}_{\text{Normal}}, \mu) \underbrace{f\left(\frac{1}{\sigma^2}\right.}_{\chi^2} \left. \Big| \, \mu\right)$$

This is similar to the setting where σ^2 unknown but μ known.

$$\frac{\sum (x_i - \mu)^2 + S_0 + n_0(\mu - \beta)^2}{\sigma^2} \,\Big|\, \mu, x_1 \dots x_n \sim \chi^2_{k_0 + 1 + n}$$

The χ^2 prior (from Previous page)

We showed that the posteriors for $\mu \mid \sigma^2$ and $\sigma^2 \mid \mu$ have the same form as their corresponding prior.

Let's find the <u>unconditional</u> posterior for σ^2.

$$f\left(\frac{1}{\sigma^2} \,\middle|\, x_1 \dots x_n\right) = \int_{-\infty}^{+\infty} f\left(\frac{1}{\sigma^2}, \mu \,\middle|\, x_1 \dots x_n\right) d\mu \propto \int_{-\infty}^{+\infty} f(x_1 \dots x_n \mid \mu, \sigma^2)\, f\left(\mu, \frac{1}{\sigma^2}\right) d\mu$$

$$\int_{-\infty}^{+\infty} \left(\frac{1}{\sigma^2}\right)^{\frac{k_0+n+1}{2}-1} e^{-\frac{\sum(x_i-\mu)^2 + S_0 + n_0(\mu-\beta)^2}{2\sigma^2}} d\mu$$

Note: $\sum(x_i-\mu)^2 = S + n(\bar{x}-\mu)^2$ where $S = \sum(x_i-\bar{x})^2$

$$\left(\frac{1}{\sigma^2}\right)^{\frac{k_0+n}{2}-1} e^{-\frac{S+S_0}{2\sigma^2}} \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sigma} e^{-\frac{n(\bar{x}-\mu)^2 + n_0(\mu-\beta)^2}{2\sigma^2}} d\mu}_{\text{cte.}\; e^{-\blacksquare/2\sigma^2}}$$

Posterior for σ^2 [unconditional]

$$\frac{S'}{\sigma^2} \sim \chi^2_K$$

where:

$$K = n + K_0$$

$$\beta' = \frac{n_0\beta + n\bar{x}}{n + n_0} \qquad \text{(same } \beta' \text{ as before)}$$

$$n' = n_0 + n$$

$$S' = S + S_0 + n_0\beta^2 + n\bar{x}^2 - n'\beta'^2$$

Back to μ. What's the unconditional prior?

$$f(\mu) \propto \int_0^\infty \left(\frac{1}{\sigma^2}\right)^{\frac{k_0+1}{2}-1} e^{-\frac{n_0(\mu-\beta)^2+S_0}{2\sigma^2}} \, d\frac{1}{\sigma^2}$$

Change of variable: $t = \dfrac{n_0(\mu-\beta)^2+S_0}{2\sigma^2}$

$$f(\mu) \propto \int_0^\infty \left[\frac{2t}{n_0(\mu-\beta)^2+S_0}\right]^{\frac{k_0+1}{2}-1} e^{-t} \frac{2}{n_0(\mu-\beta)^2+S_0} \, dt$$

$$\propto \Gamma\left(\frac{k_0+1}{2}\right)\left[n_0(\mu-\beta)^2+S_0\right]^{-\frac{k_0+1}{2}}$$

$$\propto \left[1+\left(\frac{\mu-\beta}{\sqrt{S_0/n_0k_0}}\right)^2 \Big/ k_0\right]^{-\frac{k_0+1}{2}} \equiv \left(1+\frac{X^2}{K}\right)^{-\frac{K+1}{2}}$$

So $(\mu-\beta)\Big/\sqrt{S_0/n_0k_0} \sim t_{k_0}$ in the prior

We automatically conclude that.

$$\frac{\mu - \beta'}{\sqrt{S'/n'\kappa}} \sim t_{\kappa} \text{ in the posterior}$$

Finally PHEW!

Reference Priors

$$n_0 = 0 \quad S_0 = 0 \quad k_0 = -1$$

We get

$$\left(\beta' = \bar{x}\ ,\ S' = S = \sum (x_i - \bar{x})^2,\ s^2 = \frac{S}{n-1}\right)$$

Prior:

$$\left.\begin{aligned} f(\sigma^2) &\propto \frac{1}{\sigma^2} \\ f(\mu) &\propto 1 \qquad (\text{uniform in } [-\infty, \infty]) \end{aligned}\right] \text{ improper}$$

Posterior:

$$\left.\begin{aligned} \frac{S}{\sigma^2} &\sim \chi^2_{n-1} \\ \frac{\mu - \bar{x}}{s/\sqrt{n}} &\sim t_{n-1} \end{aligned}\right] \text{ what we knew}$$

What happens if:

$$x_i \mid \mu, \sigma^2 \sim N(\mu, \sigma^2)$$

$$\frac{S_0}{\sigma^2} \sim \chi^2_{k_0}$$

$$\mu \sim N(\beta, \tau^2)$$

(Semi-conjugate prior)

Later

Skip Last page of Note 8.