

## The t-distribution (Student)

- $Z \sim N(0, 1)$
- $V \sim \chi^2_K$
- $Z$  and  $V$  are independent

Then  $t = \frac{Z}{\sqrt{V/K}}$  satisfies  $f(t) \propto \left(1 + \frac{t^2}{K}\right)^{-\frac{K+1}{2}}$

[ look up  
constant ]

Observation: when  $K$  is large, this is almost Normal.

$$\lim_{K \rightarrow \infty} \left(1 + \underbrace{\frac{K+1}{K} \cdot \frac{t^2}{K+1}}_1\right)^{-\frac{K+1}{2}} = e^{-t^2/2}$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n \rightarrow e^x$$

$$E[t] = 0 \text{ if } K > 1 \quad \text{Var}[t] = \begin{cases} \frac{K}{K-2} & K > 2 \\ \infty & 1 < K \leq 2 \end{cases}$$

Fisher proved the following for Normal samples

- $(n-1)s^2/\sigma^2 \sim \chi_{n-1}^2$        $\frac{S}{\sigma^2} \sim \chi_{n-1}^2$  where  $S = \sum(x_i - \bar{x})^2$  (\*)
- $\bar{x}$  and  $s^2$  are independent (only Normal satisfies this)
- (We know)  $\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$

So  $\frac{\bar{x} - \mu}{s/\sqrt{n}} \sim t_{n-1}$  ( $t$ -distributed with degree  $n-1$ )

Remark: The text in Note 8 contains proofs for the first two facts when  $n=2$ .

(\*) This is different than previous definition  $S = \sum(x_i - \mu)^2$  when  $\mu$  was known.

## Two teams revisited (unknown $\sigma^2$ )

$$\frac{(n-1) s_x^2}{\sigma_x^2} \sim \chi_{n-1}^2$$

$$\frac{(m-1) s_y^2}{\sigma_y^2} \sim \chi_{m-1}^2$$

So  $\frac{(n-1) s_x^2}{\sigma_x^2} + \frac{(m-1) s_y^2}{\sigma_y^2} \sim \chi_{n+m-2}^2$  [sum of ind.  $\chi^2$ ]

From before

$$\frac{\bar{x} - \bar{y} - (\mu_x - \mu_y)}{\sqrt{\sigma_x^2/n + \sigma_y^2/m}} \sim N(0, 1)$$

Assuming  $\sigma_x^2 = \sigma_y^2$  [though both are unknown], testing  $\mu_x = \mu_y$

we get

$$\frac{\bar{x} - \bar{y}}{\sqrt{\left(\frac{1}{n} + \frac{1}{m}\right) \frac{(n-1)s_x^2 + (m-1)s_y^2}{n+m-2}}} \sim t_{n+m-2}$$

[Note 8 has an example]

## Conjugate Prior

$x_i | \mu, \sigma^2 \sim N(\mu, \sigma^2)$  are i.i.d.

$$\frac{s_0}{\sigma^2} \sim \chi_{k_0}^2$$

$$\mu | \sigma^2 \sim N(\beta, \frac{\sigma^2}{n_0})$$

\* Making  $\mu$  &  $\sigma$  independent does not lead to conjugate

We want to prove posterior has the same form.

$$\frac{s'}{\sigma^2} | x_1, \dots, x_n \sim \chi_k^2 \quad (\text{Find } s' \text{ and } k)$$

$$\mu | \sigma^2, x_1, \dots, x_n \sim N(\beta', \frac{\sigma^2}{n'}) \quad (\text{Find } \beta' \text{ and } n')$$

First, let's find  $\mu | \sigma^2, x_1, \dots, x_n$

$$f(\mu | x_1, \dots, x_n, \sigma^2) \propto \underbrace{f(x_1, \dots, x_n | \mu, \sigma^2)}_{\text{Normal}} \cdot \underbrace{f(\mu | \sigma^2)}_{\text{Normal}}$$

This is the same setting where  $\mu$  is unknown but  $\sigma^2$  known

$$\mu | \sigma^2, x_1, \dots, x_n \sim N \left( \frac{\sigma^2 \beta/n + \sigma^2/n_0 \bar{x}}{\sigma^2/n + \sigma^2/n_0}, \frac{\sigma^2/n \sigma^2/n_0}{\sigma^2/n + \sigma^2/n_0} \right)$$

$$\mu | \sigma^2, x_1, \dots, x_n \sim N \left( \frac{n_0 \beta + n \bar{x}}{n+n_0}, \frac{\sigma^2}{n+n_0} \right) = N \left( \beta', \frac{\sigma^2}{n'} \right)$$

- $\beta'$  is weighted average of  $\beta$  and  $\bar{x}$
- $n$  = # observations
- $n_0$  = # "observations" in prior.

What about  $\sigma^2$ ? Let's start with

$$f(\mu, \frac{1}{\sigma^2}) \propto \underbrace{\frac{1}{\sigma} e^{-\frac{n_0(\mu-\beta)^2}{2\sigma^2}}}_{f(\mu|\sigma^2)} \cdot \underbrace{\left(\frac{1}{\sigma^2}\right)^{\frac{k_0}{2}-1} e^{-\frac{S_0}{2\sigma^2}}}_{f(\frac{1}{\sigma^2})}$$
$$= \left(\frac{1}{\sigma^2}\right)^{\frac{k_0+1}{2}-1} e^{-\frac{n_0(\mu-\beta)^2 + S_0}{2\sigma^2}}$$

Observe that  $f\left(\frac{1}{\sigma^2} | \mu\right) \propto f\left(\mu, \frac{1}{\sigma^2}\right)$

$$\text{so } \frac{S_0 + n_0(\mu-\beta)^2}{\sigma^2} | \mu \sim \chi^2_{k_0+1}$$

(Another way of  
looking at prior  
for  $\sigma^2$ )

Let's find  $\frac{1}{\sigma^2} | \mu, x_1, \dots, x_n$

$$f\left(\frac{1}{\sigma^2} | x_1, \dots, x_n, \mu\right) \propto \underbrace{f(x_1, \dots, x_n | \sigma^2, \mu)}_{\text{Normal}} \underbrace{f\left(\frac{1}{\sigma^2} | \mu\right)}_{\chi^2}$$

?

Normal

$\chi^2$

This is similar to the setting where  $\sigma^2$  unknown but  $\mu$  known.

$$\frac{\sum (x_i - \mu)^2 + S_0 + n_0(\mu - \beta)^2}{\sigma^2} \mid \mu, x_1, \dots, x_n \sim \chi^2_{k_0 + 1 + n}$$

The  $\chi^2$  prior (from previous page)

We showed that the posteriors for  $\mu | \sigma^2$  and  $\sigma^2 | \mu$  have the same form as their corresponding prior.

Let's find the unconditional posterior for  $\sigma^2$ .

$$f\left(\frac{1}{\sigma^2} \mid x_1, \dots, x_n\right) = \int_{-\infty}^{+\infty} f\left(\frac{1}{\sigma^2}, \mu \mid x_1, \dots, x_n\right) d\mu \propto \int_{-\infty}^{+\infty} f(x_1, \dots, x_n \mid \mu, \sigma^2) f(\mu, \frac{1}{\sigma^2}) d\mu$$

$$\int_{-\infty}^{+\infty} \left(\frac{1}{\sigma^2}\right)^{\frac{k_0+n+1}{2}-1} e^{-\frac{\sum(x_i-\mu)^2 + S_0 + n_0(\mu-\beta)^2}{2\sigma^2}} d\mu$$

Note:  $\sum(x_i-\mu)^2 = S + n(\bar{x}-\mu)^2$  where  $S = \sum(x_i-\bar{x})^2$

$$\left(\frac{1}{\sigma^2}\right)^{\frac{k_0+n}{2}-1} e^{-\frac{S+S_0}{2\sigma^2}} \int_{-\infty}^{+\infty} \frac{1}{\sigma} e^{-\frac{n(\bar{x}-\mu)^2 + n_0(\mu-\beta)^2}{2\sigma^2}} d\mu$$

$\underbrace{\quad}_{\text{cte.}} e^{-\frac{\square}{2\sigma^2}}$

Posterior for  $\sigma^2$  [unconditional]

$$\frac{S'}{\sigma^2} \sim \chi^2_K$$

where:  $K = n + k_0$

$$\beta' = \frac{n_0 \beta + n \bar{x}}{n + n_0} \quad (\text{same } \beta' \text{ as before})$$

$$n' = n_0 + n$$

$$S' = S + S_0 + n_0 \beta^2 + n \bar{x}^2 - n' \beta'^2$$

Back to  $\mu$ . What's the unconditional prior?

$$f(\mu) \propto \int_0^{\infty} \left( \frac{1}{\sigma^2} \right)^{\frac{K_0+1}{2}-1} e^{-\frac{n_0(\mu-\beta)^2 + S_0}{2\sigma^2}} d\frac{1}{\sigma^2}$$

Change of variable:  $t = \frac{n_0(\mu-\beta)^2 + S_0}{2\sigma^2}$

$$f(\mu) \propto \int_0^{\infty} \left[ \frac{2t}{n_0(\mu-\beta)^2 + S_0} \right]^{\frac{K_0+1}{2}-1} e^{-t} \frac{2}{n_0(\mu-\beta)^2 + S_0} dt$$

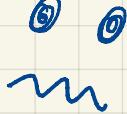
$$\propto \left[ \left( \frac{K_0+1}{2} \right) \left[ n_0(\mu-\beta)^2 + S_0 \right] \right]^{-\frac{K_0+1}{2}}$$

$$\propto \left[ 1 + \left( \frac{\mu-\beta}{\sqrt{S_0/n_0 K_0}} \right)^2 / K_0 \right]^{-\frac{K_0+1}{2}} \equiv \left( 1 + \frac{x^2}{K} \right)^{-\frac{K+1}{2}}$$

$$S_0 \quad (\mu-\beta) / \sqrt{S_0/n_0 K_0} \sim t_{K_0} \text{ in the prior}$$

We automatically conclude that.

$$\frac{\mu - \beta'}{\sqrt{s'/n'k}} \sim t_k \text{ in the posterior}$$

Finally  PHEW!

## Reference Priors

$$n_0 = 0 \quad S_0 = 0 \quad K_0 = -1$$

We get

Prior:

$$f(\sigma^2) \propto \frac{1}{\sigma^2}$$

$$f(\mu) \propto 1 \quad (\text{Uniform in } [-\infty, \infty])$$

improper

Posterior:

$$\frac{S}{\sigma^2} \sim \chi^2_{n-1}$$

$$\frac{\mu - \bar{x}}{S/\sqrt{n}} \sim t_{n-1}$$

what we knew

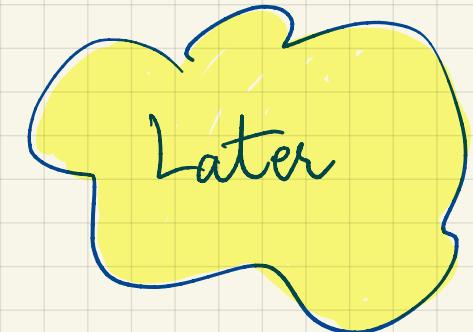
What happens if:

$$x_i | \mu, \sigma^2 \sim N(\mu, \sigma^2)$$

$$\frac{s_0}{\sigma^2} \sim \chi_{k_0}^2$$

$$\mu \sim N(\beta, \gamma^2)$$

(Semi-Conjugate prior)



Skip Last page of Note 8.