Binomial $\mid p$

prior

$f(p)$

$\propto p^{\alpha-1}(1-p)^{\beta-1}$

observe

k success

and $n-k$ failures

$f(p \mid S_n = k)$

$\propto p^{(k+\alpha)-1}(1-p)^{(n-k+\beta)-1}$

Beta density

Beta Priors

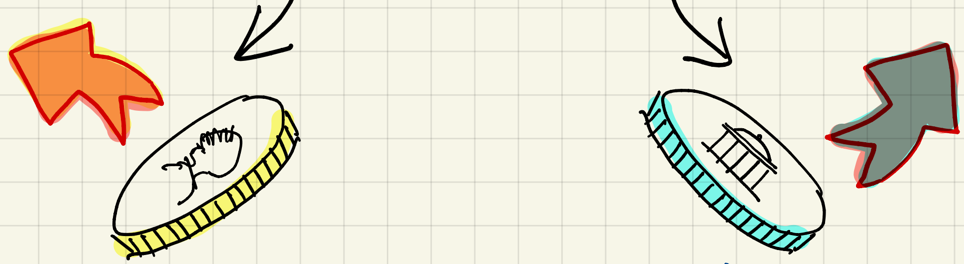
Recall the Beta density

$$f(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, p^{\alpha-1}(1-p)^{\beta-1}$$

Assume the prior for the prob. of Heads is a mixture of two Beta priors.

$$f(p) = q\, Be(\alpha,\beta) + (1-q)\, Be(\beta,\alpha)$$

$(\alpha > \beta)$



$Be(\alpha,\beta)$ $\quad$ $Be(\beta,\alpha)$

We observe k heads in n tosses. What is the posterior of p?

Posterior due to $Be(\alpha, \beta)$: $Be(\alpha + k, \beta + n - k)$

Posterior due to $Be(\beta, \alpha)$: $Be(\beta + k, \alpha + n - k)$

Mixture: Recall

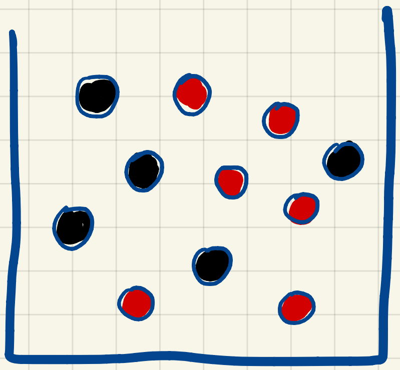
$$\frac{q(k)}{1 - q(k)} = \frac{q}{1-q} \frac{\int \binom{n}{k} p^k (1-p)^{n-k} Be(\alpha, \beta)\, dp}{\int \binom{n}{k} p^k (1-p)^{n-k} Be(\beta, \alpha)\, dp}$$

$$= \frac{q}{1-q} \frac{\int p^{k+\alpha-1} (1-p)^{n-k+\beta-1}\, dp}{\int p^{k+\beta-1} (1-p)^{n-k+\alpha-1}\, dp}$$

$$\int p^{\alpha-1} (1-p)^{\beta-1}\, dp = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$= \frac{q}{1-q} \frac{\Gamma(k+\alpha)\Gamma(n-k+\beta)}{\Gamma(k+\beta)\Gamma(n-k+\alpha)}$$

Polya's Urn



b black balls

r red balls

Repeat:

- pick a ball
- Look at color
- return it with c new balls of the same color

When $c=0$, drawings are equivalent to independent Bernoullis with $p = \frac{b}{b+r}$

When $c \neq 0$, the Bernoullis are not independent.

e.g. the second pick depends on the first one.

Examples:

$$P(BBR): \frac{b}{b+r} \cdot \frac{b+c}{b+r+c} \cdot \frac{r}{b+r+2c}$$

$$P(BRB): \frac{b}{b+r} \cdot \frac{r}{b+r+c} \cdot \frac{b+c}{b+r+2c}$$

$$P(RBB): \frac{r}{b+r} \cdot \frac{b}{b+r+c} \cdot \frac{b+c}{b+r+2c}$$

Order of B, R in sequence does not matter.

Let $S_n = \#$ black balls seen in n trials $= X_1 + X_2 + \dots + X_n$

$P(S_n = k) = \binom{n}{k} p_{n,k}$ [all sequences with k black have same probability]

(As in Binomial)

Interesting fact:

$$P(X_i = 1) = \frac{b}{b+r} \text{ still!}$$

[and this does not mean that X_i is independent of $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n$]

$$P(X_i=1) = \sum_{\substack{X_1 \cdots X_{i-1} \\ X_{i+1} \cdots X_n}} P(X_1 \ldots X_{i-1}\ B\ X_{i+1} \ldots X_n)$$

$$= \sum_{\substack{X_1 \cdots X_{i-1} \\ X_{i+1} \cdots X_n}} P(B \ldots X_{i-1}\ X_1\ X_{i+1} \ldots X_n) = P(X_1 = B)$$

$$= \frac{b}{b+r}$$

Example: $P(X_2=1) = P(B\overset{\downarrow}{B}) + P(R\overset{\downarrow}{B})$

$$= P(B)\cdot P(B|B) + P(R)\cdot P(B|R)$$

$$= \frac{b}{b+r}\cdot\frac{b+c}{b+r+c} + \frac{r}{b+r}\cdot\frac{b}{b+r+c}$$

$$= \frac{b}{b+r}\underbrace{\left[\frac{b+c}{b+r+c} + \frac{r}{b+r+c}\right]}_{1}$$

Another interesting fact.

$$P(X_i = x \mid X_j = y) = \frac{P(X_i = x, X_j = y)}{P(X_j = y)}$$

$$= \frac{P(X_i = y, X_j = x)}{P(X_i = y)}$$

$$= P(X_j = x \mid X_i = y)$$

Back to $p_{n,k}$: $P(\underbrace{BB\ldots B}_{k}\ \underbrace{RR\ldots\ldots R}_{n-k})$

$$p_{n,k} = \frac{\prod_{i=1}^{k}[b+(i-1)c]\prod_{i=1}^{n-k}[r+(i-1)c]}{\prod_{i=1}^{n}[b+r+(i-1)c]}$$

$$= \frac{\prod_{i=1}^{k}\left(\frac{b}{c}+i-1\right)\prod_{i=1}^{n-k}\left(\frac{r}{c}+i-1\right)}{\prod_{i=1}^{n}\left(\frac{b}{c}+\frac{r}{c}+i-1\right)}$$

$\frac{b}{c}\left(\frac{b}{c}+1\right)\left(\frac{b}{c}+2\right)\cdots$

$$= \frac{\dfrac{\Gamma\left(\frac{b}{c}+k\right)}{\Gamma\left(\frac{b}{c}\right)}\cdot\dfrac{\Gamma\left(\frac{r}{c}+n-k\right)}{\Gamma\left(\frac{r}{c}\right)}}{\dfrac{\Gamma\left(\frac{b}{c}+\frac{r}{c}+n\right)}{\Gamma\left(\frac{b}{c}+\frac{r}{c}\right)}}$$

$$P(S_n=k)=\frac{\Gamma(\frac{b}{c}+\frac{r}{c})\,\Gamma(k+\frac{b}{c})\,\Gamma(n-k+\frac{r}{c})\,\Gamma(n+1)}{\Gamma(\frac{b}{c})\Gamma(\frac{r}{c})\,\Gamma(k+1)\,\Gamma(n-k+1)\,\Gamma(n+\frac{b}{c}+\frac{r}{c})}$$

where $\quad = \binom{n}{k} = \dfrac{n!}{k!(n-k)!} = \dfrac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}$

Stirling's approximation $\Gamma(n) \approx \sqrt{2\pi/n}\left(\frac{n}{e}\right)^n$ for large n.

$$\frac{\Gamma(n+a)}{\Gamma(n+b)} \approx n^{a-b} \quad (n \text{ large})$$

Assume $k\to\infty$, but $k \leq nx$ where $0<x<1$

$$P(S_n=k) \approx \frac{\Gamma(\frac{b}{c}+\frac{r}{c})}{\Gamma(\frac{b}{c})\Gamma(\frac{r}{c})}\; k^{\frac{b}{c}-1}(n-k)^{\frac{r}{c}-1}\, n^{1-\frac{b}{c}-\frac{r}{c}}$$

$$= \frac{\Gamma(\frac{b}{c}+\frac{r}{c})}{\Gamma(\frac{b}{c})\Gamma(\frac{r}{c})}\; x^{\frac{b}{c}-1}(1-x)^{\frac{r}{c}-1}\, n^{-1}$$

Proof of $\dfrac{\Gamma(n+a)}{\Gamma(n+b)} \to n^{a-b}$ for large n.

$$\frac{\Gamma(n+a)}{\Gamma(n+b)} \approx \sqrt{\frac{n+b}{n+a}}\ \frac{(n+a)^{n+a}}{(n+b)^{n+b}}\ e^{b-a}$$

$$\approx 1 \cdot \frac{n^{n+a}\left(1+\frac{a}{n}\right)^{n+a}}{n^{n+b}\left(1+\frac{b}{n}\right)^{n+b}}\ e^{b-a}$$

$$= n^{a-b} \cdot \frac{\left(1+\frac{a}{n}\right)^{a}}{\left(1+\frac{b}{n}\right)^{b}}\ \frac{e^{a}}{e^{b}}\ e^{b-a}$$

$$\approx n^{a-b} \cdot \frac{1+a^2/n}{1+b^2/n} \cdot 1 = n^{a-b}.$$

$$P\left(\frac{S_n}{n} \leq x\right) = P\left(\frac{S_n}{n} = 0\right) + P\left(\frac{S_n}{n} = \frac{1}{n}\right) + \cdots + P\left(\frac{S_n}{n} = \frac{\lfloor nx \rfloor}{n}\right)$$

Now, $$\lim_{n\to\infty} \frac{1}{n}\Big[\underbrace{P\left(\frac{S_n}{n}=0\right) + \cdots + P\left(\frac{S_n}{n} = \frac{\lfloor nx \rfloor}{n}\right)}_{P\left(\frac{S_n}{n} \leq x\right)}\Big] = \int_0^x P\left(\frac{S_n}{n} = u\right) du$$

$$P\left(\frac{S_n}{n} \leq x\right) = n\int_0^x P(S_n = nu)\, du$$

Since $nu \to \infty$, replace k with nu in the expression for $P(S_n = k)$

we get $$P\left(\frac{S_n}{n} \leq x\right) = \int_0^x \frac{\Gamma\left(\frac{b}{c}+\frac{r}{c}\right)}{\Gamma\left(\frac{b}{c}\right)\Gamma\left(\frac{r}{c}\right)} u^{\frac{b}{c}-1} (1-u)^{\frac{r}{c}-1} du$$

Therefore $\frac{S_n}{n} \sim Be\left(\frac{b}{c}, \frac{r}{c}\right)$ as n goes to infinity.

Let $Y = \lim_{n\to\infty} \frac{S_n}{n}$.

It's also know that

$$P(S_n = k \mid Y = p) = \binom{n}{k} p^k (1-p)^{n-k}$$

In other words S_n is Binomial with parameters k, n, p conditioned on $Y = p$.