Reversible Markov Chain

$$P(X_n = i \mid X_{n+1} = j, X_{n+2} = k, \ldots)$$

$$= \frac{P(X_n = i, X_{n+1} = j, X_{n+2} = k, \ldots)}{P(X_{n+1} = j, X_{n+2} = k, \ldots)}$$

$$= \frac{P(X_n = i)\, P(X_{n+1} = j \mid X_n = i)\, P(X_{n+2} = k \mid X_{n+1} = j, X_n = i) \ldots}{P(X_{n+1} = j)\, P(X_{n+2} = k \mid X_{n+1} = j) \ldots}$$

$$= \frac{P(X_n = i)\; P_{ij} \cdot \cancel{P_{jk}} \cdot \cancel{\ldots}}{P(X_{n+1} = j)\, \cancel{P_{jk}}\, \cancel{\ldots}} = \frac{P(X_n = i)\, P_{ij}}{P(X_{n+1} = j)}$$

$$P(X_n = i \mid X_{n+1} = j, X_{n+2}, X_{n+3} \dots)$$

$$= P(X_n = i \mid X_{n+1} = j) = \frac{P(X_n = i) P_{ij}}{P(X_{n+1} = j)}$$

Let's assume n large, we reach stationary distribution

$$P(X_n = i) = \pi_i$$

$$P(X_{n+1} = j) = \pi_j$$

$$P(X_n = i \mid X_{n+1} = j) = \frac{\pi_i}{\pi_j} P_{ij}$$

Markov chain is reversible if

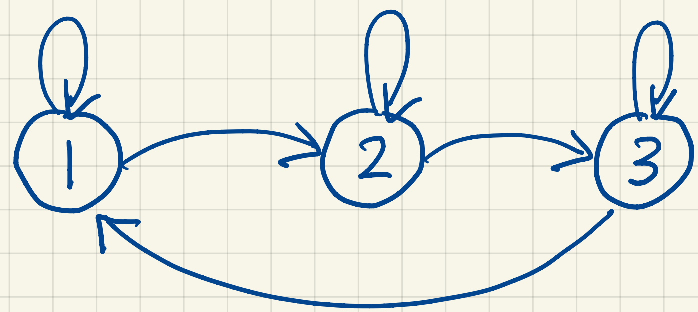
$$\forall\, i,j: \quad \underbrace{P(X_n = i \mid X_{n+1} = j)}_{\text{"backward"}} = \underbrace{P(X_{n+1} = i \mid X_n = j)}_{\text{"forward"}}$$

$$\frac{\pi_i}{\pi_j} P_{ij} = P_{ji}$$

$$\boxed{\forall\, i,j. \quad \pi_i P_{ij} = \pi_j P_{ji}}$$

(Detailed balance equation)

Example when detailed balance is not satisfied



irreducible

aperiodic

$$P_{ij} > 0 \Rightarrow P_{ji} = 0 \qquad (i \neq j)$$

$$\pi_i \underbrace{P_{ij}}_{>0} = \pi_j \underbrace{P_{ji}}_{=0} \qquad \text{(Not Reversible)}$$

What's the big deal about detailed balance

$$\forall\, i,j. \quad \pi_i P_{ij} = \pi_j P_{ji} \quad ?$$

If we can find $\pi = (\pi_1, \pi_2, \dots, \pi_K)$ that satisfies above, then π is stationary.

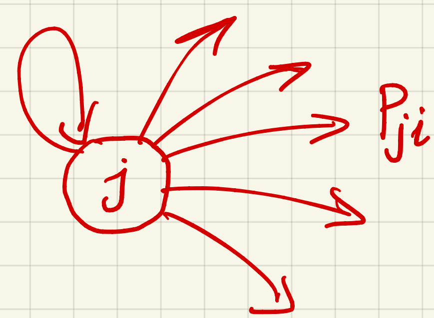
Proof:

$$\pi_i P_{ij} = \pi_j P_{ji}$$

$$\underbrace{\sum_i \pi_i P_{ij}}_{\text{Prob of being in state } j \text{ in the next step}} = \sum_i \pi_j P_{ji}$$

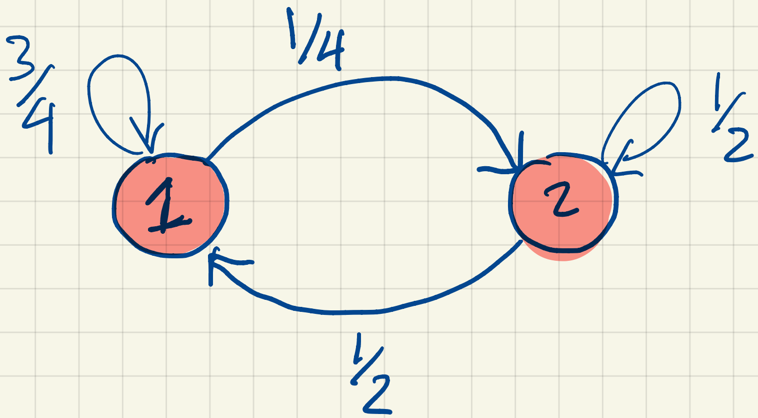
$$= \pi_j \underbrace{\sum_i P_{ji}}_{1}$$

$$= \pi_j$$



Same as. $\pi P = \pi$

Example:



$\pi_1 P_{12} = \pi_2 P_{21}$

$\pi_1 \frac{1}{4} = \pi_2 \frac{1}{2}$

$\pi_1 + \pi_2 = 1$

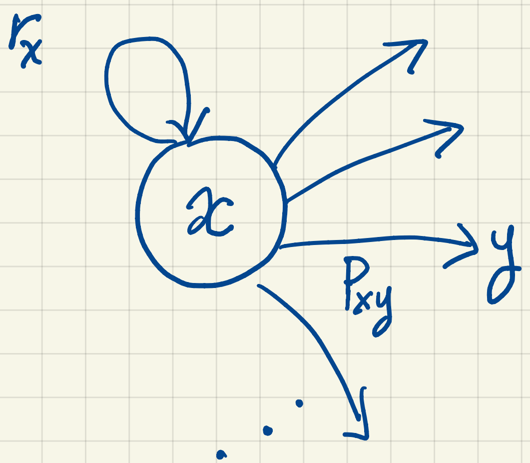
$$\left.\begin{aligned} \pi_1 &= 2\pi_2 \\ \pi_1 + \pi_2 &= 1 \end{aligned}\right| \Rightarrow 3\pi_2 = 1 \Rightarrow \pi_2 = \frac{1}{3}$$

$$\pi_1 = \frac{2}{3}$$

Summary:

$$\left(\exists \pi . \left(\forall i,j . \left(\pi_i P_{ij} = \pi_j P_{ji}\right)\right)\right) \Longrightarrow \pi \text{ is stationary}$$

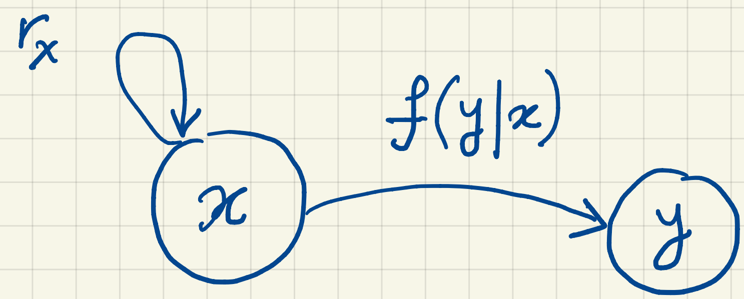
Discrete Markov Chain



$$r_x + \sum_{y \neq x} P_{xy} = 1$$

$$\sum_{y} P_{xy} = 1$$

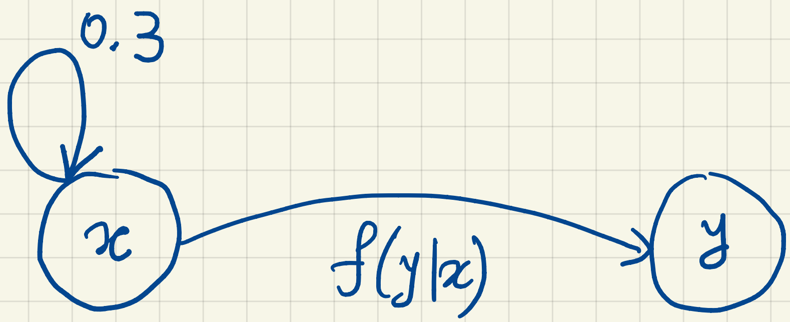
Continuous Markov chain



Conditioned on x, y has density $f(y|x)$

$$r_x + \underbrace{\int f(y|x)\, dy}_{1 - r_x} = 1$$

Example:



$$f(y|x) = 0.7 \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(y-x)^2}{2\sigma^2}}$$

"Given x, y is Normal with mean $= x$ and variance σ^2"

Why do we allow a non-zero prob. of staying in the state, i.e. $P(X_{n+1} = i \mid X_n = i) > 0$? (Later)