Examples of Baye's Rule:

posterior $P(A|B) = \dfrac{P(B|A)\,P(A)}{P(B)}$ prior

Binary

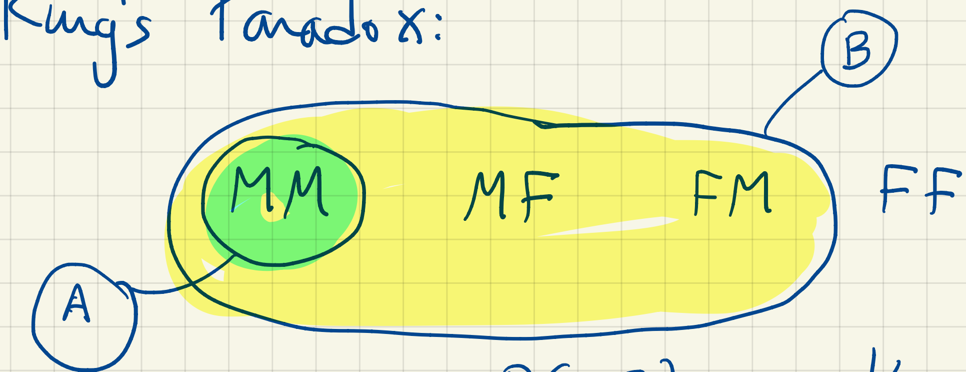
$A \wedge A^c$ (A's complement)

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} \qquad \left(P(A^c) = 1 - P(A)\right)$$

Multiple event: $A_1, A_2, \dots, A_k$

$$P(A_i|B) = \frac{P(B|A_i)\,P(A_i)}{\sum_{j=1}^{k} P(B|A_j)\,P(A_j)} \qquad \sum_{i=1}^{k} P(A_i) = 1$$

King's Paradox:



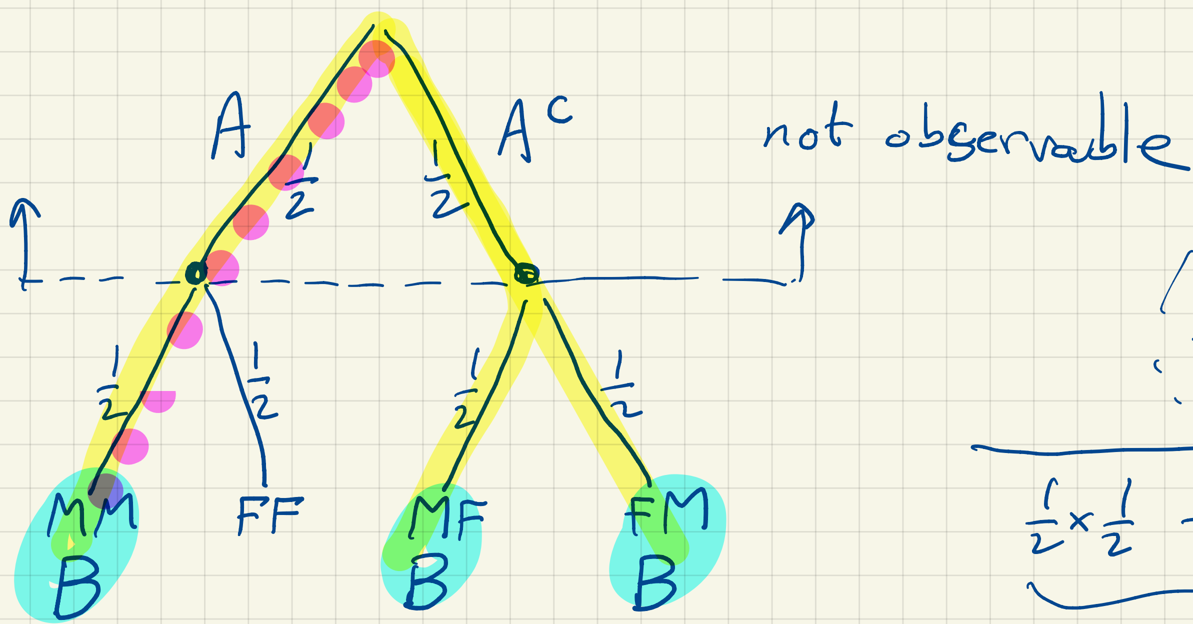
Listing outcomes & identifying sets

$$P(A|B) = \frac{P(A,B)}{P(B)} = \frac{1/4}{3/4} = 1/3$$

A : Same gender (not observable)

B : there is at least one male (observable)

Prob. Tree



Desired path

$$\frac{\frac{1}{2}\times\frac{1}{2}}{\underbrace{\frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2}}_{\text{all paths}}} = \frac{1}{3}$$

Medical Test:

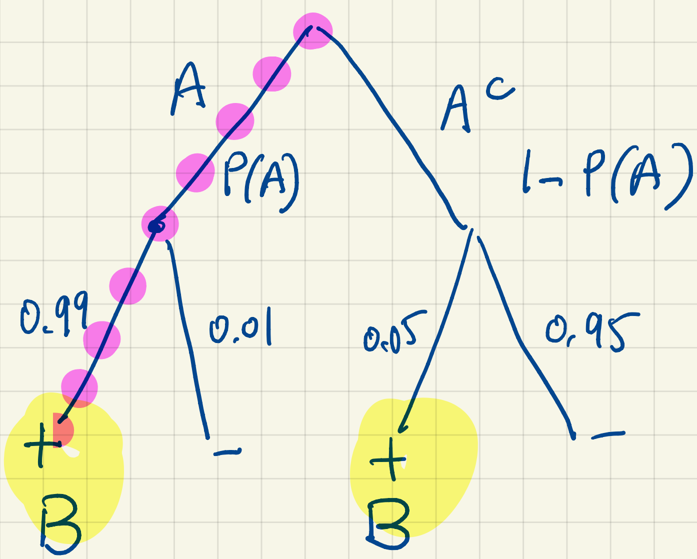
A: Patient has Disease

B: Patient tests Positive

$P(A|B) = ?$ $\qquad P(B|A) = 0.99 \qquad P(B|A^c) = 0.05$

Patient tested positive. What's the prob he has disease?

$(\neq 0.99)$



$$\frac{0.99\,P(A)}{0.99\,P(A) + 0.05\,[1-P(A)]}$$

$$= \frac{0.99\,P(A)}{0.94\,P(A) + 0.05} < \frac{P(A)}{0.05}$$

$P(A) = 0.0001 \Rightarrow P(A|B)$ is very small

"Symbolic": Use Bayes Formula & what we know about the probabilities (No pictures).

- $$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

$$= \frac{0.99\,P(A)}{0.99\,P(A) + 0.05(1-P(A))}$$

- King's Paradox:

$$P(\text{Same}\,|\,1M) = \frac{P(1M|\text{Same})\,P(\text{Same})}{P(1M|\text{Same})\,P(\text{Same}) + P(1M|\text{Diff})\cdot P(\text{Diff})}$$

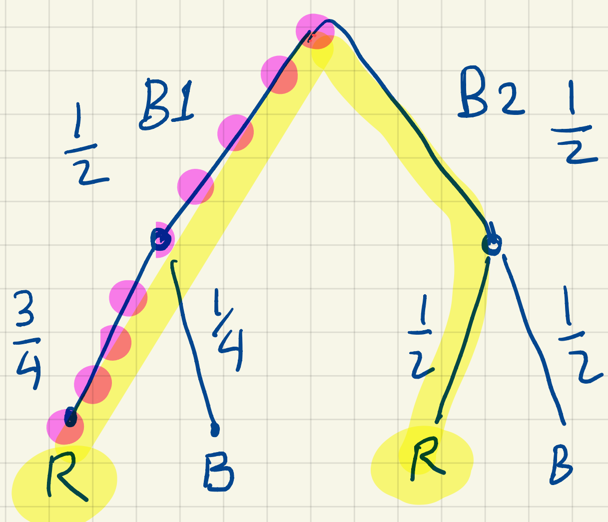
$$= \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2} + 1\cdot\frac{1}{2}} = \frac{1}{3}$$

B1	B2
30 red 10 black	10 red 10 black

1) Pick a bin at random with equal prob.

2) Pick a ball from it.

We observe a Red Ball. What's the prob. it's coming from Bin 1.



$$\frac{\frac{1}{2} \times \frac{3}{4}}{\frac{1}{2} \times \frac{3}{4} + \frac{1}{2} \times \frac{1}{2}} = \frac{3}{5}$$

Symbolic:

R: Red B: Black B1: Bin 1 B2: Bin 2

$$P(B1|R) = \frac{P(R|B1)\,P(B1)}{P(R|B1)\,P(B1) + P(R|B2)\,P(B2)}$$

$$= \frac{3/4 \cdot 1/2}{\frac{3}{4}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}} = \frac{3}{5}$$

1) Pick a Bin at random with prob $\frac{1}{2}$

2) Pick two balls sequentially without replacement.

Let's say we observe RB (Red then Black)

What's the prob. it's Bin 1

$$P(B1 \mid RB) = \frac{P(RB \mid B1)\, P(B1)}{P(RB \mid B1) P(B1) + P(RB \mid B2)\, P(B2)}$$

"Symbolic"

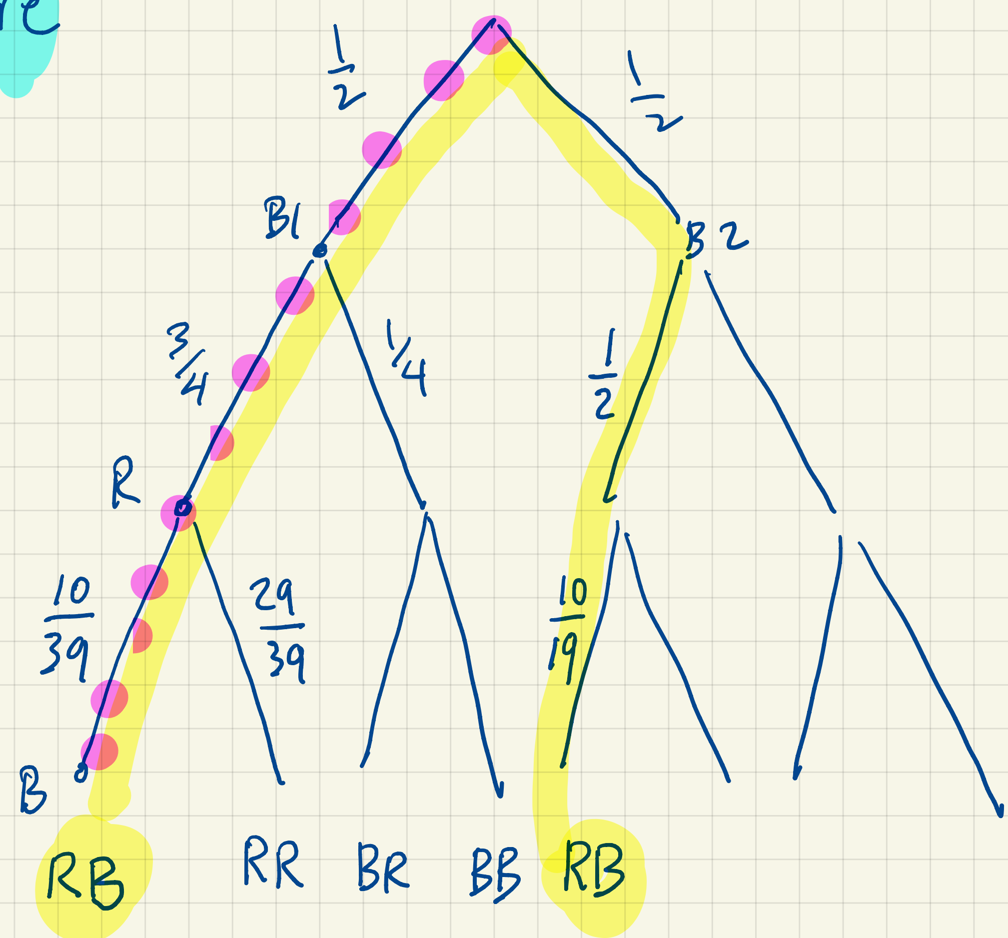
$$P(R \mid B1)\, \underbrace{P(B \mid R, B1)}_{\neq P(B \mid B1)} \cdot P(B1)$$

$$= \overline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

$\ldots$

$$= \frac{\frac{30}{40} \times \frac{10}{39} \times \frac{1}{2}}{\frac{30}{40} \times \frac{10}{39} \times \frac{1}{2} + \frac{10}{20} \times \frac{10}{19} \times \frac{1}{2}} = 0.4222\ldots.$$

Tree structure



Sequential Bayes:

Original Prior: $P(B_1) = P(B_2) = \frac{1}{2}$

Updated Prior (posterior after Red) $P(B_1) = \frac{3}{5}$ $P(B_2) = \frac{2}{5}$

$$P(B_1 | B, R) = \frac{P(B | B_1, R) \cdot P(B_1 | R)}{P(B|B_1, R)\, P(B_1 | R) + P(B | B_2, R)\, P(B_2 | R)}$$

$$= \frac{P(B | B_1)\, P(B_1)}{P(B|B_1)\, P(B_1) + P(B|B_2) P(B_2)}$$ (Just looking at blue)

Don't use

$$\frac{\frac{10}{40} \times \frac{1}{2}}{\frac{10}{40} \times \frac{1}{2} + \frac{10}{20} \times \frac{1}{2}}$$

(Just looking at blue makes it more familiar but probabilities must still account for history)

$$= \frac{10/39 \cdot 3/5}{\frac{10}{39} \times \frac{3}{5} + \frac{10}{19} \times \frac{2}{5}} = 0.4222\ldots$$

Sequential Bayes same as one shot Bayes.

$$\overset{n}{P}(x) = P(x \mid x_1 \dots x_n) = \frac{P(x_1 \dots x_n \mid x)\, P(x)}{P(x_1 \dots x_n)}$$

$$= \frac{P(x_1 \dots x_{n-1} \mid x)\, P(x_n \mid x_1 \dots x_{n-1}, x)\, P(x)}{P(x_1 \dots x_{n-1})\, P(x_n \mid x_1 \dots x_{n-1})}$$

$$= \frac{P(x_n \mid x_1 \dots x_{n-1}, x)\; P(x \mid x_1 \dots x_{n-1})}{P(x_n \mid x_1 \dots x_{n-1})}$$

$$= \frac{P(x_n \mid x_1 \dots x_{n-1}, x)\; \overset{n-1}{P}(x)}{P(x_n \mid x_1 \dots x_{n-1})}$$

New prior