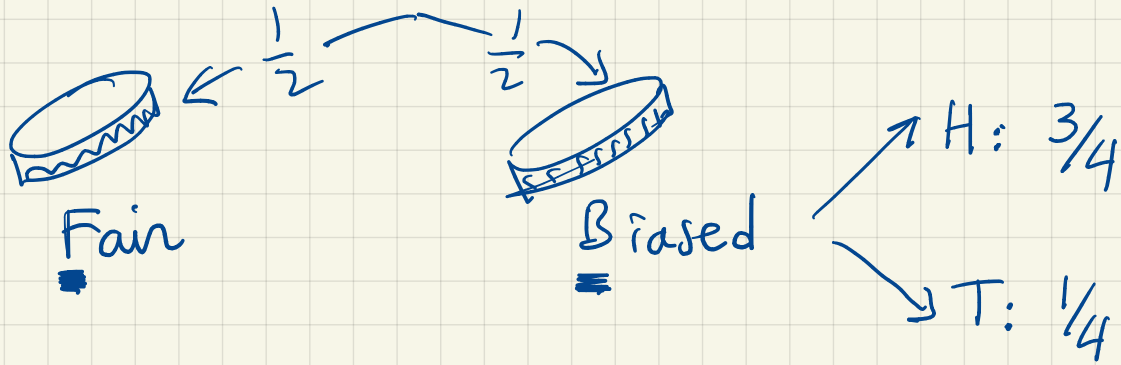
More Bayes Rule Examples:



$$P(F|H) = \frac{P(H|F)\,P(F)}{P(H)} = \frac{P(H|F)\,P(F)}{P(H|F)P(F) + P(H|B)P(B)}$$

posterior

$$= \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2} + \frac{3}{4}\cdot\frac{1}{2}} = \frac{2}{2+3} = \frac{2}{5}$$

: The likelihood, determined by physics of system

: Prior, either given or some model assumed

: Denominator, most complicated, resolve it by conditioning (it becomes a sum)

$$P(F \mid T, \mathbf{H}) = \frac{P(T \mid F, \mathbf{H})\, P(F \mid \mathbf{H})}{P(T \mid \mathbf{H})}$$

$P(F \mid H)$: we have it from before

$P(T \mid F, H)$: equal $P(T \mid F)$ because coin tosses are independent conditioned on coin

$P(T \mid H)$: There is no conditioning on coin, so these are not independent.

$$P(T \mid H) \neq \frac{1}{2} \cdot P(F) + \frac{1}{4} P(B) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2}$$

$$P(T \mid H) = \underbrace{P(T \mid F, H)}_{P(T \mid F)} P(F \mid H) + \underbrace{P(T \mid B, H)}_{P(T \mid B)} P(B \mid H)$$

$$= \frac{1}{2} \cdot \frac{2}{5} + \frac{1}{4} \cdot \frac{3}{5}$$

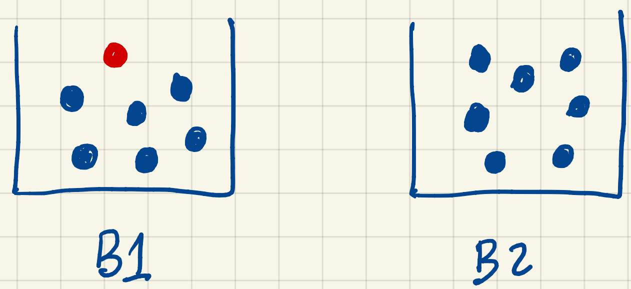
$$P(F \mid T, H) = \frac{\frac{1}{2} \cdot \frac{2}{5}}{\frac{1}{2} \cdot \frac{2}{5} + \frac{1}{4} \cdot \frac{3}{5}} = \frac{4}{4+3} = \frac{4}{7}$$

One shot:

$$P(F \mid HT) = \frac{P(HT/F)\, P(F)}{P(HT/F) P(F) + P(HT/B) P(B)}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{2} \times \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{4} \times \frac{1}{2}} = \frac{4}{7}$$

Games using Bayes.



Each bin contain n balls

B_i: event that bin i has the winning ball

$$P(B_1) = P(B_2) = \frac{1}{2}$$

b_i : event that I draw winning ball from Bin i

We draw a ball from Bin 1, and we see that it's **NOT** the winning ball. What's the prob. that B1 contains the winning ball?

$$P(B1 \mid b_1^c) = \frac{P(b_1^c \mid B1)\, P(B1)}{P(b_1^c \mid B1)\, P(B1) + P(b_1^c \mid B2)\, P(B2)}$$

$$= \frac{\left(1 - \frac{1}{n}\right) \cdot 1/2}{\left(1 - \frac{1}{n}\right) \cdot \frac{1}{2} + 1 \cdot 1/2} = \frac{n-1}{n-1+n} = \frac{n-1}{2n-1} < \frac{1}{2}$$

$$P(B2 \mid b_1^c) = \frac{n}{2n-1} > \frac{1}{2}$$

<u>Bin 1</u>   <u>Bin 2</u>

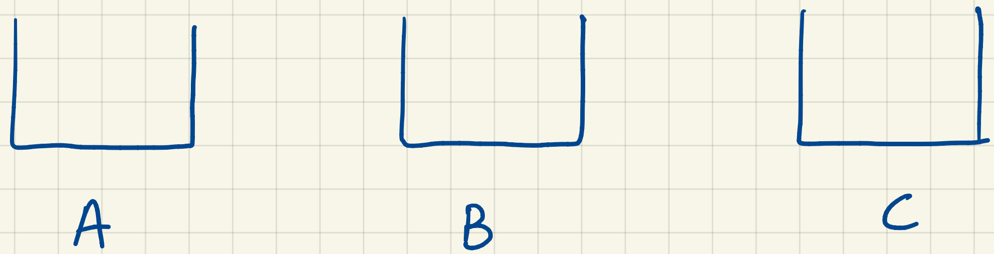
<u>Replacement</u> (ball goes back)

$$\frac{1}{n} \cdot \frac{n-1}{2n-1} < \frac{1}{n} \cdot \frac{n}{2n-1}$$

<u>No Replacement</u> (ball is thrown away)

$$\frac{1}{n-1} \cdot \frac{n-1}{2n-1} = \frac{1}{n} \cdot \frac{n}{2n-1}$$

Monty Hall Problem:



- One of the boxes contain a prize, with equal prob $= \frac{1}{3}$
- we choose a box at random with prob $= \frac{1}{3}$
- the host opens one of the remaining boxes and shows you it's empty. (Host knows everything)
  (randomly when multiple choices are available.)

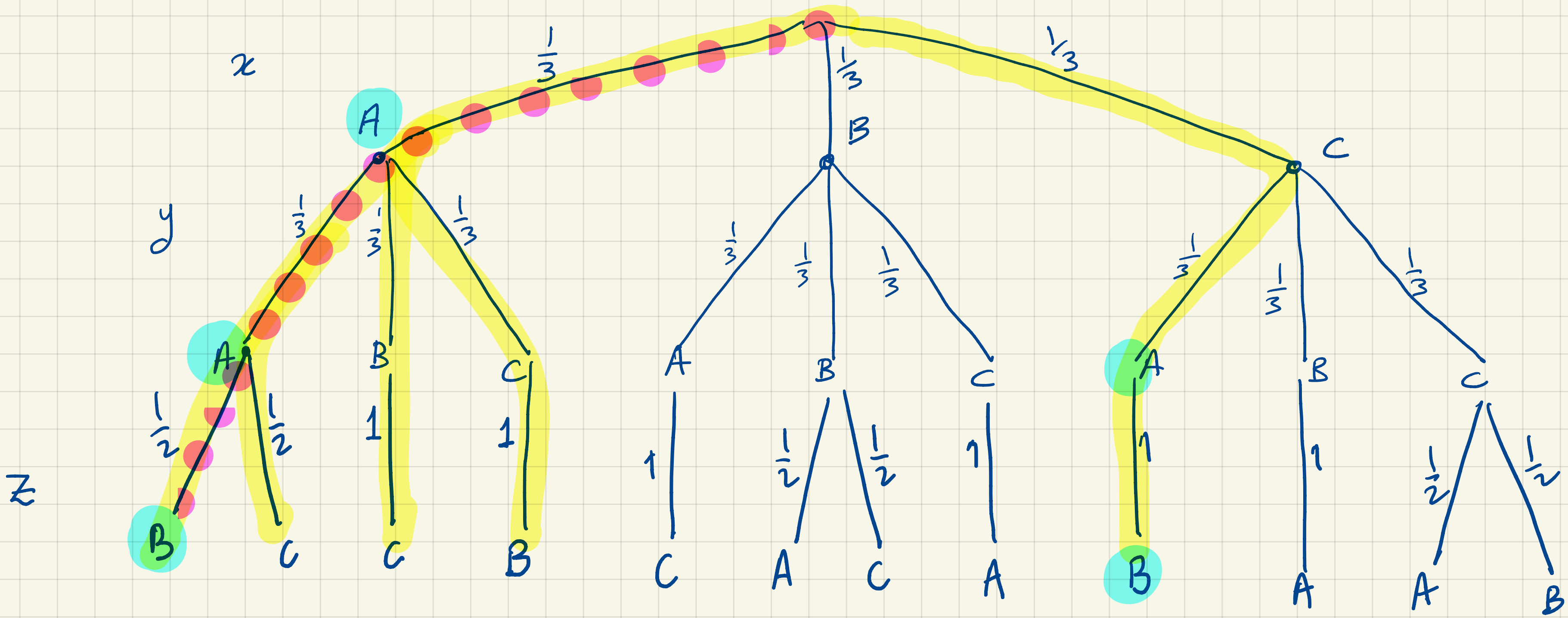
What is the prob. that you have the winning box?

x: the winning box

y: the box we choose

z: the opened box.

$P(x \mid y, z)$ for example: $P(x = A \mid y = A, z = B)$



$$\frac{\frac{1}{3} \times \frac{1}{3} \times \frac{1}{2}}{\frac{1}{3} \times \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{3} \times 1} = \frac{1/2}{1/2 + 1} = \frac{1}{3}$$

$$P(x|y,z) = \left(0, \frac{1}{3}, \frac{2}{3}\right)$$ (x & y dependent when conditioned on z)

$$P(x|y) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$ (x & y are independent not conditioned on z)

$$P(x|z) = \left(0, \frac{1}{2}, \frac{1}{2}\right)$$

$$P(x|y,z) = \begin{cases} 0 & x = z \\ \frac{1}{3} & x = y \\ \frac{2}{3} & \text{otherwise} \end{cases}$$

$$P(x|y) = \frac{1}{3} \quad \forall \ x$$

$$P(x|z) = \begin{cases} 0 & x = z \\ \frac{1}{2} & x = y \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

$$P(x|y,z) = \frac{P(y,z|x)\,P(x)}{\sum_x P(y,z|x)P(x)} = \frac{P(z|y,x)\,P(y|x)\,P(x)}{\sum_x P(z|y,x)\underbrace{P(y|x)}_{1/3}\underbrace{P(x)}_{1/3}}$$

$$= \frac{1/9\ P(z|y,x)}{1/9\ \sum_x P(z|y,x)} = \frac{P(z|y,x)}{\sum_x P(z|y,x)}$$

Example:

$$P(x=A|y=A, z=B) = \frac{P(z=B|y=A, x=A)}{P(z=B|y=A,x=A)+P(z=B|y=A,x=B)+P(z=B|y=A,x=C)}$$

$$= \frac{1/2}{1/2 + 0 + 1} = \frac{1}{3}$$

Similarly:

$P(x=B|y=A, z=B)=0$, $P(x=C|y=A, z=B)= \frac{2}{3}$