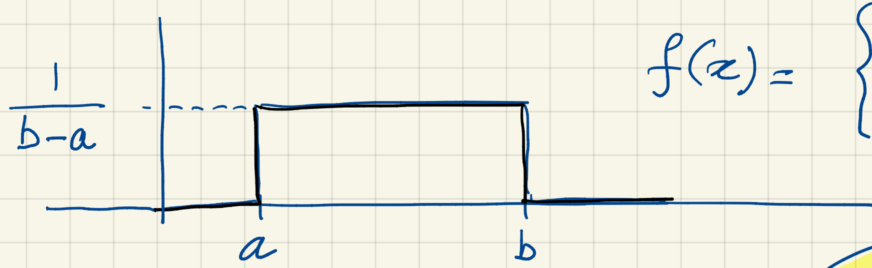
Continuous Uniform random variable



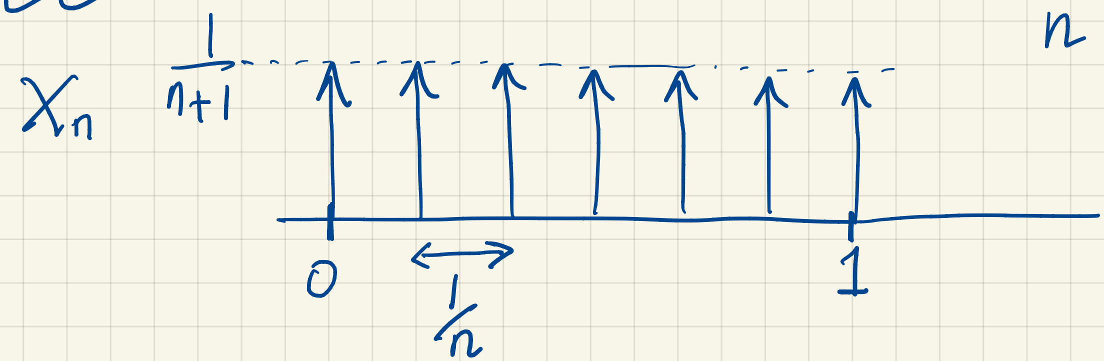
$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

But how do we physically obtain such R.V. What's the underlying setting?

Special case : $a=0, \quad b=1$

Discrete Case:

n : large number



X_n takes values $x \in \left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n}{n}\right\}$ with prob. $\frac{1}{n+1}$ each

$$P(a \le X_n \le b) = \frac{1}{n+1}\left(\#\text{ outcomes in } [a,b]\right)$$

$$\approx \frac{1}{n+1} \frac{b-a}{1/n} = \frac{n}{n+1}(b-a)$$

$$\lim_{n\to\infty} P(a \le X_n \le b) \longrightarrow b-a = \int_a^b \underbrace{f(x)}_{\text{density}}\,dx$$

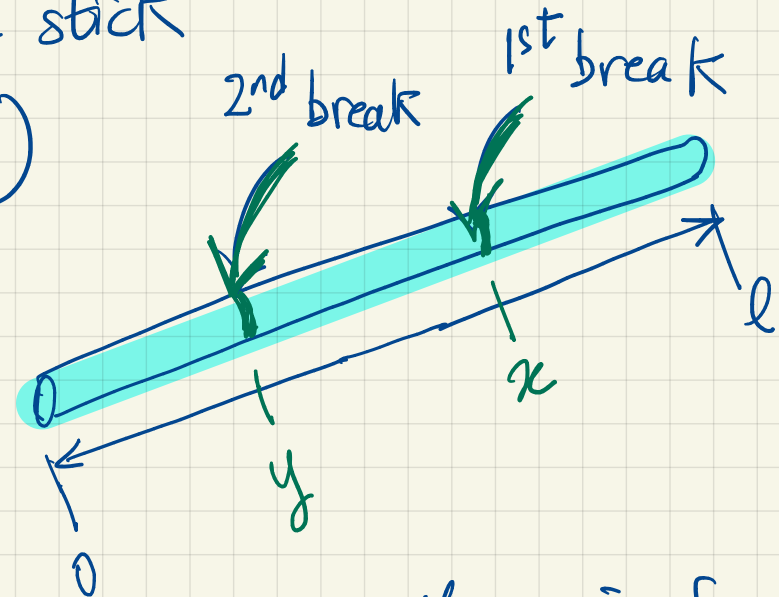
$$\text{set } f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

We are not saying that prob. mass function converges to the density.

The Probability $P(a \le X_n \le b)$ is converging to the $\int_a f(x)dx$

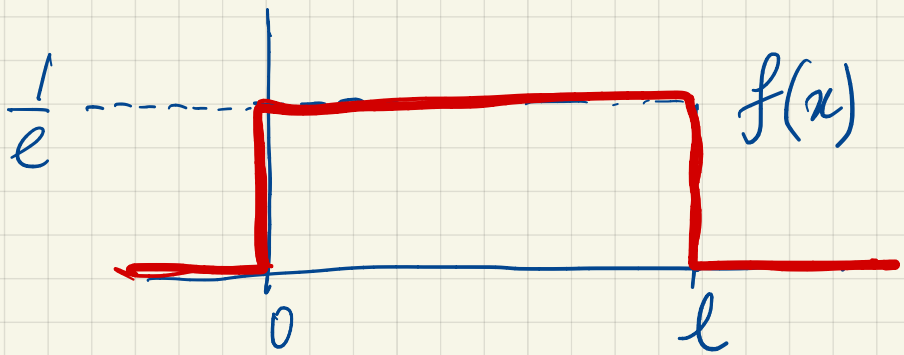
Random Variable converges <u>in distribution</u> to cont. uniform

Breaking a stick (twice)



x is uniform in $[0, \ell]$

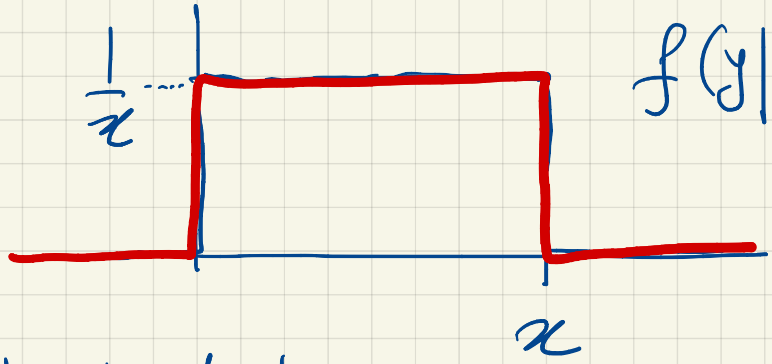
$$f(x) = \begin{cases} \frac{1}{\ell} & 0 \leq x \leq \ell \\ 0 & \text{otherwise} \end{cases}$$



y is uniform in $[0, x]$

<u>conditioned on x</u>

$$f(y|x) = \begin{cases} \frac{1}{x} & 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$



$f(y|x) \qquad 0 \leq y \leq x$

Let's try to find:

$f(x,y)$ ✓ (Joint density)

$f(y)$ ✓ (marginal density)

$E[x]$

$E[y]$

$f(x|y)$

Forget the setting, here's the math!

- x is uniform in $[0, \ell]$
- Conditioned on x, y is uniform in $[0, x]$

$$f(x) = \begin{cases} \dfrac{1}{\ell} & \underline{0 \le x \le \ell} \\ 0 & \end{cases}$$

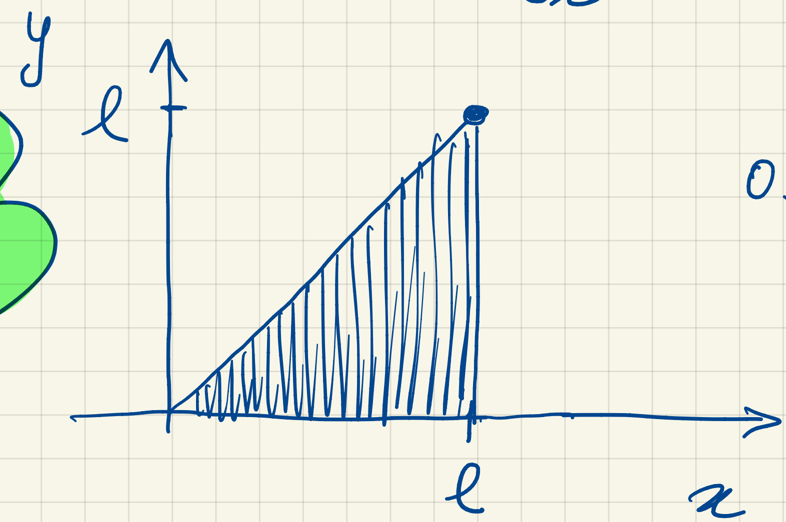
$$f(y|x) = \begin{cases} \dfrac{1}{x} & \underline{0 \le y \le x} \\ 0 & \end{cases}$$

$$f(x,y) = f(y|x)\,f(x)$$

$P(A,B) = P(B|A)\,P(A)$

$$= \frac{1}{x}\,\frac{1}{\ell} = \frac{1}{\ell x}$$ whenever it's not zero

Domain

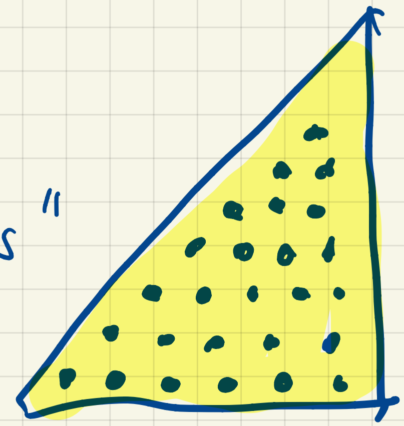


$0 \leq y \leq x \leq \ell$

$$f(x,y) = \begin{cases} \dfrac{1}{\ell x} & 0 \leq y \leq x \leq \ell \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^{l}\int_y^{l} \underline{f(x,y)}\,dx\,dy = 1$$

Go over all the "points"



for $y \leftarrow 0$ to l
    for $x \leftarrow y$ to l
        ⋮

$$\int_0^{l}\int_0^{x} \underline{f(x,y)\,dy}\,dx = 1$$

for $x \leftarrow 0$ to l
    for $y \leftarrow 0$ to x
        ⋮

$$\int_0^{\ell}\int_0^{x} \underbrace{\frac{1}{\ell x}}\, dy\, dx = \int_0^{\ell} \frac{y}{\ell x}\Bigg|_0^{x} dx = \int_0^{\ell} \underbrace{\frac{1}{\ell}}_{f(x)} dx = \frac{x}{\ell}\Bigg|_0^{\ell} = 1$$

$$\int_0^{\ell}\int_y^{\ell} \frac{1}{\ell x}\, dx\, dy = \int_0^{\ell} \frac{\ln x}{\ell}\Bigg|_y^{\ell} dy = \int_0^{\ell} \underbrace{\frac{1}{\ell} \ln \frac{\ell}{y}}_{f(y)} dy$$

$$f(y) = \int \underbrace{f(y|x) f(x)}_{f(x,y)} dx$$

$$P(B) = \sum_i \underbrace{P(B|A_i) P(A_i)}_{P(B, A_i)}$$

$$= \int_y^\ell \frac{1}{\ell x} dx = \frac{\ln x}{\ell} \Big|_y^\ell$$

$$= \frac{\ln \ell - \ln y}{\ell} = \frac{\ln \ell/y}{\ell}$$

$$f(y) = \begin{cases} \dfrac{\ln \ell/y}{\ell} & 0 \leq y \leq \ell \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_0^\ell x f(x)\,dx = \frac{\ell}{2} \quad (\text{uniform})$$

In general if X is uniform in $[a, b]$

$$E[X] = \frac{a+b}{2} \quad (\text{middle})$$

$$E[y] = \int_0^\ell y f(y)\,dy = \int_0^\ell y \frac{\ln \ell/y}{\ell}\,dy = \cdots\cdots ?$$

<u>Nested Expection</u>

$$E[Y \mid X = x] = \frac{x}{2}$$

$$E[Y] = \underset{X}{E}\left[\frac{x}{2}\right] = \frac{1}{2} E[x] = \frac{1}{2} \cdot \frac{\ell}{2} = \frac{\ell}{4}$$

$$E[XY] = ?$$

$$E[XY \mid X=x] = E[xY \mid X=x]$$

constant

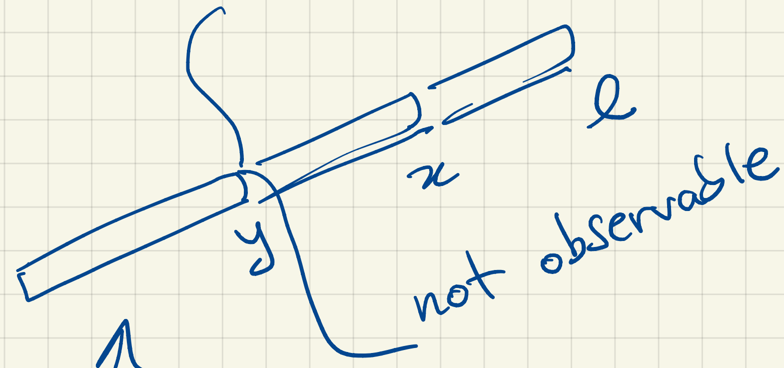
$$= x \underbrace{E[Y \mid X=x]}_{x \cdot \frac{x}{2} = \frac{x^2}{2}}$$

$$E[XY] = E\left[\frac{x^2}{2}\right] = \int_0^{\ell} \frac{x^2}{2} f(x)\,dx = \int_0^{\ell} \frac{x^2}{2\ell}\,dx$$

$$= \frac{x^3}{6\ell}\Bigg|_0^{\ell} = \frac{\ell^2}{6}$$

$$E[XY] = \int_0^{\ell}\int_y^{\ell} xy\, f(x,y)\, dx\, dy$$

$$= \int_0^{\ell}\int_y^{\ell} \cancel{x}y\, \frac{1}{\cancel{x}\ell}\, dx\, dy = \text{----} \quad [\text{easy}]$$



looking at end result.

$$f(x|y) = \frac{f(y|x)f(x)}{f(y)} = \frac{f(x,y)}{f(y)}$$

$$= \frac{\frac{1}{\ell x}}{\frac{\ln \ell/y}{\ell}} = \frac{1}{x \ln \ell/y} \qquad y \leqslant x \leqslant \ell$$

$$f(x|y) = \begin{cases} \dfrac{1}{x \ln \ell/y} & y \leqslant x \leqslant \ell \\ 0 & \text{otherwise} \end{cases}$$

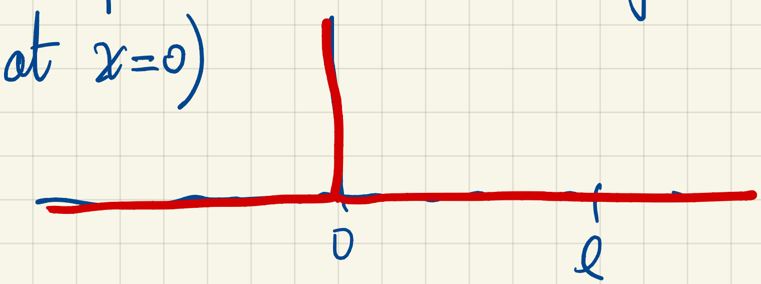
Remark: $$\int_y^\ell \frac{1}{x \ln \ell/y}\, dx = \left.\frac{\ln x}{\ln \ell/y}\right|_y^\ell = \frac{\ln \ell - \ln y}{\ln \ell/y} = 1$$

$$f(x|y) = \begin{cases} \dfrac{1}{x \ln \ell/y} & y \le x \le \ell \\ 0 & \text{otherwise} \end{cases}$$

$y = 0$: $\quad f(x|y=0) = \dfrac{1}{x \cdot \infty} \qquad 0 \le x \le \ell$

it's zero everywhere except at $x=0$ it's infinite

(all the mass is at $x=0$)



$y = \ell$: $\quad f(x|y=\ell) = \dfrac{1}{x \cdot 0} = \infty \qquad \ell \le x \le \ell$

(all the mass is at $x=\ell$)