Other random variables.

So far:

Discrete	Continuous
uniform	uniform
Binomial	
geometric	Exponential
Bernoulli	
Poisson	Normal (Gaussian)

Strategy: 1) Underlying physical experiment
2) Limiting procedure

Poisson (Approximation to Binomial)

Binomial

$$X = X_1 + X_2 + \cdots + X_n \quad (\text{i.i.d})$$

$$X_i = \begin{cases} 1 & p \\ 0 & 1-p \end{cases}$$

independent & identically distributed

$$P(X=k) = b(k,n,p) = \binom{n}{k} p^k (1-p)^{n-k}$$

Setting: $n \longrightarrow \infty$, $p \longrightarrow 0$, $np \longrightarrow \lambda$

Practically: n large, p is small, np moderate

$$p = \frac{\lambda}{n} \qquad (\lambda \text{ is constant})$$

$$n \to \infty$$

$$b(k, n, p) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \frac{n!}{k!\,(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{\overbrace{n(n-1)(n-2)\cdots(n-k+1)}^{k \text{ terms}}}{k!\,\underbrace{n^k}_{1}} \lambda^k \frac{\left(1-\frac{\lambda}{n}\right)^n}{\underbrace{\left(1-\frac{\lambda}{n}\right)^k}_{1}}$$

$$\lim_{n\to\infty} b(k, n, p) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= P(k, \lambda)$$

k	$P(k,\lambda)$	$b(k,n,p)$
0	0.2541	0.2537
1	0.3481	0.3484
$\vdots$	$\vdots$	$\vdots$
5	0.0102	0.0101
6	0.0023	0.0023

$n = 500$

$p = \frac{1}{365}$

$np = \lambda = 1.3699$

$P(k,\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$ is a valid probability mass function

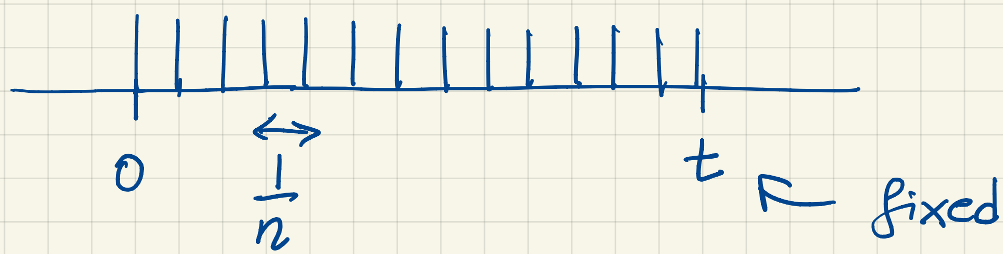
$$\sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}_{e^{\lambda}} = e^{-\lambda} \cdot e^{\lambda} = 1.$$

If x is Poisson distributed:

$$E[X] = \lambda$$

$$\sigma_X^2 = \lambda$$

Events occur over time



- Divide interval of time $[0,t]$ into small intervals of size $\frac{1}{n}$ (n is large)

- Event occurs in a small interval with probability p that is small

- During interval $[0,t]$, I have $\approx nt$ small interval (trials in a binomial R.V)

$$b(k, nt, p)$$

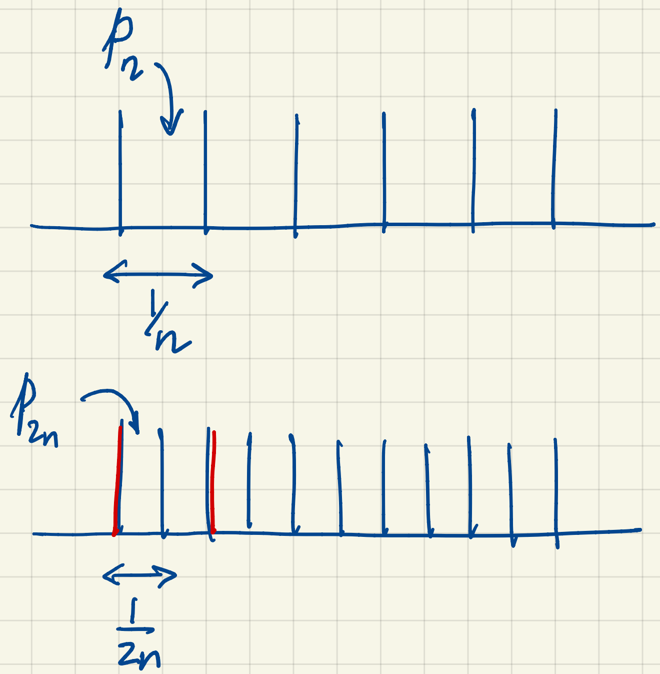
\# events in $[0,t]$ · \# trials · prob. of event

- Let's assume that $np \to \lambda$ (np does not vanish)
  (np does not go to ∞)

- $ntp \to \lambda t$

$$P(k \text{ events in } [0,t]) \approx \frac{(\lambda t)^k e^{-\lambda t}}{k!} \quad \text{(Poisson approx)}$$

\# events in $[0,t]$
is a discrete quantity $\in \{0, 1, 2, 3, \dots\}$



$$1 - p_n = (1 - p_{2n})(1 - p_{2n})$$

$$1 - p_n = 1 - 2p_{2n} + p_{2n}^2$$

$$p_n = 2p_{2n} - p_{2n}^2$$

$$np_n = 2np_{2n} - np_{2n}^2$$

so $\quad np_n < 2np_{2n}$

So np_n does not go to zero

np_n does not go to infinity either because this means we expect infinitely many event in any small interval t.

let T = time until the first event

(continuous quantity)

$$P(T \leqslant t) = ?$$

example: $t = 10$ sec

What's the prob. that the first event occurs within 10 sec ?

This means within 10 sec I have one or more events.

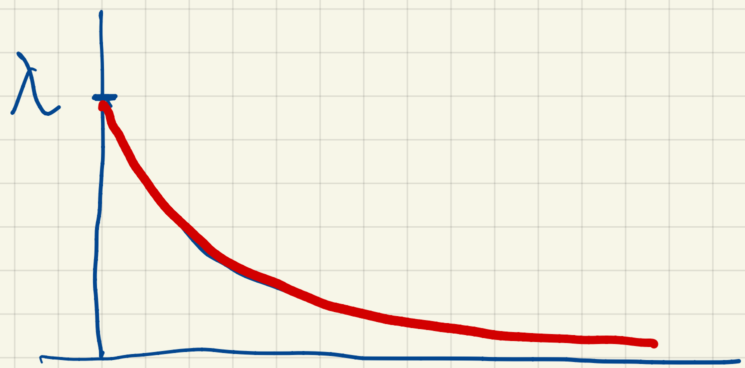
$$P(k=1) + P(k=2) + P(k=3) + \cdots\cdots = 1 - P(k=0)$$

For any t,

$$P(T \leqslant t) = 1 - \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = 1 - e^{-\lambda t}$$

$$P(T \leq t) = \int_0^t f_T(z)\,dz = 1 - e^{-\lambda t}$$

$$\boxed{f_T(t) = \lambda e^{-\lambda t}}$$ exponential density.



If X is exponentially distributed $\boxed{f(x) = \lambda e^{-\lambda x}}$

then $E[x] = \frac{1}{\lambda}$

$$\sigma_X^2 = \frac{1}{\lambda^2}$$

$$P(T \geq t+\tau \mid \underline{\underline{T \geq \tau}})$$

I waited for τ and nothing happened

what's the prob. that I have to wait for an

<u>additional</u> time t

$$P(T \geq t+\tau \mid T \geq \tau) = \frac{\overbrace{P(T \geq \tau \mid T \geq t+\tau)}^{1} P(T \geq t+\tau)}{P(T \geq \tau)}$$

$$= \frac{e^{-\lambda(t+\tau)}}{e^{-\lambda\tau}} = e^{-\lambda t}$$

$$= P(T \geq t) \qquad \text{[memoryless]}$$

Waiting for the additional time t, has the same prob.

as waiting for t starting at time 0.

Bayes rule with densities

$$f(x|y) = \frac{f(y|x)\, f(x)}{\int f(y|x)\, f(x)\, dx}$$

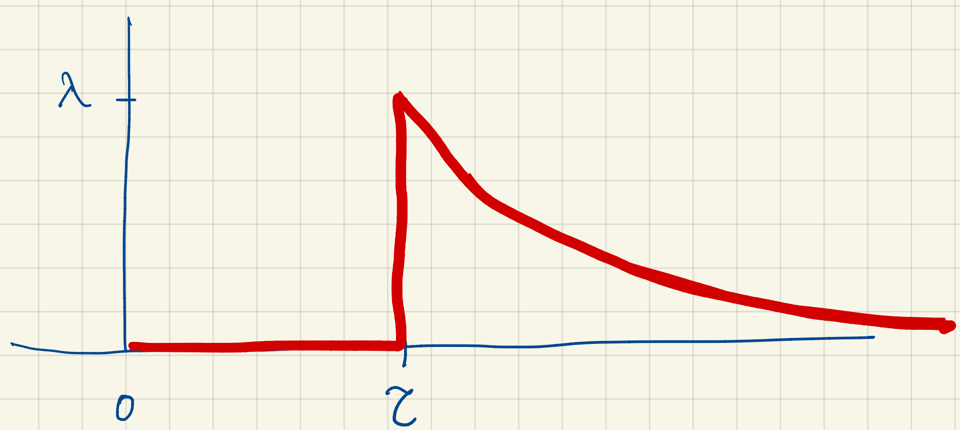
$$f_T(t \mid \underbrace{T \geq \tau}_{\text{Event}}) = \frac{P(T \geq \tau \mid T = t)\, f_T(t)}{\int_0^\infty P(T \geq \tau \mid T = t)\, f_T(t)\, dt}$$

the denominator is not 0 if $t < \tau$ [see below]

$$P(T \geq \tau \mid T = t) = \begin{cases} 1 & \text{if } t \geq \tau \\ 0 & \text{otherwise} \end{cases}$$

$$f_T(t \mid T \geq \tau) = \frac{\mathbb{1}_{t \geq \tau} \cdot f_T(t)}{\int_0^\tau \underbrace{P(T \geq \tau \mid T = t)}_{0} f(t)\,dt + \int_\tau^\infty \underbrace{P(T \geq \tau \mid T = t)}_{1} f(t)\,dt}$$

$$= \frac{\mathbb{1}_{t \geq \tau} f(t)}{\int_\tau^\infty f(t)\,dt = P(T \geq \tau)} = \begin{cases} \dfrac{\lambda e^{-\lambda t}}{e^{-\lambda \tau}} = \lambda e^{-\lambda (t-\tau)} & t \geq \tau \\ 0 & \text{otherwise} \end{cases}$$



In general

$$f(x \mid E) = \frac{P(E \mid X=x)\, f(x)}{\int_{-\infty}^{+\infty} P(E \mid X=x)\, f(x)\, dx}$$

$$P(E \mid x) = \frac{f(x \mid E)\, P(E)}{f(x \mid E) P(E) + f(x \mid E^c)\, [1 - P(E)]}$$