Problem 0: Readings
Read note 2 from the course web site.

Problem 1: The dishonest casino
A casino has two coins, one of them is fair, the other is not. This can be modeled as follows:

\[ P(H|\text{fair}) = P(T|\text{fair}) = 1/2 \]

\[ P(H|\text{biased}) = 1 - P(T|\text{biased}) = p \]

Note that this is slightly different than the classical approach where one would express the probabilities as follows:

\[ P(H) = P(T) = 1/2 \quad \text{coin is fair} \]
\[ P(H) = 1 - P(T) = p \quad \text{coin is biased} \]

This is typical with Bayesian analysis when the event of being fair or biased can itself be considered as a probabilistic outcome rather than a given. With probability \( q \), the casino will use the biased coin. You get to observe the result of the flip; however, you do not know which coin is actually being used.

(a) Let \( p = 0.9 \) and \( q = 0.1 \), a Bayesian approach is to compute the probability that a particular coin has been used after observing several outcomes. This should give an indication to which coin is more likely to have influenced the observation. Let’s do this for two outcomes. Find:

\[ P(F|HH) \quad P(F|HT) \quad P(F|TH) \quad P(F|TT) \]

where \( F \) stands for fair. Do this using two ways: 1) one shot, and 2) two steps, and confirm that the result is the same.

Solution:
I will do one of them, and leave the others because the same approach works for the solution.

\[ P(F|HH) = \frac{P(HH|F)P(F)}{P(HH|F)P(F) + P(HH|B)P(B)} = \frac{0.25 \times 0.9}{0.25 \times 0.9 + 0.81 \times 0.1} = 0.735. \]
Using sequential Bayes rule, we have:

\[ P(F|H) = \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|B)P(B)} = \frac{0.5 \times 0.9}{0.5 \times 0.9 + 0.9 \times 0.1} = 0.8333 \]

Now using the above result as the new prior, we have for the second H:

\[ P(F|H) = \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|B)P(B)} = \frac{0.5 \times 0.8333}{0.5 \times 0.8333 + 0.9 \times 0.1666} = 0.735 \]

(b) Is it true that:
\[ P(F|H) = P(F|HH) + P(F|HT) \]

Explain.

**Solution:** This is generally not true; for instance, both \( P(F|HH) \) and \( P(F|HT) \) could be high and, therefore, their sum greater than 1. We cannot split the conditioning this way.

**Problem 2: Forgetful Monty Hall**

Make sure you understand why it is always better to switch in the Monty Hall problem described in class. Think about the problem from different angles. Keep it active in your mind for some time.

Now assume that Monty Hall forgot where the prize was. In the game, he will simply open one of the remaining boxes at random with equal probability. Consider the case where an empty box is revealed. Show that there is no benefit in switching. Show your work symbolically and using a tree.

**Solution:** Let us consider \( P(x = A|y = A, z = B, B \text{ is empty}) \). This is equal to (using Bayes’ rule):

\[
\begin{align*}
P(y = A, z = B, B \text{ is empty}|x = A)P(x = A) \\
&= P(y = A, z = B, B \text{ is empty}|x = A)P(x = A) + P(y = A, z = B, B \text{ is empty}|x = B)P(x = B) + P(y = A, z = B, B \text{ is empty}|x = C)P(x = C) \\
&= \frac{1/3 \cdot 1/2 \cdot 1 \cdot 1/3}{1/3 \cdot 1/2 \cdot 1 \cdot 1/3 + 1/3 \cdot 1/2 \cdot 0 \cdot 1/3 + 1/3 \cdot 1/2 \cdot 1 \cdot 1/3} = 1/2
\end{align*}
\]

**Problem 3: Find the winning ball**

There are \( k \) bins, and \( kn \) balls. Only one ball is winning. The balls are randomly placed in the bins, in such a way that each bin receives exactly \( n \) balls. Define the following events for \( i = 1 \ldots k, m = 1 \ldots n \):

- \( B_i \): bin \( i \) contains the winning ball
- \( b_{i,m} \): picking the \( m^{th} \) ball from bin \( i \) reveals the winning one

At this point, we have the following: \( P(B_i) = \frac{1}{k} \) and \( P(b_{i,1}|B_i) = \frac{1}{n} \).
Therefore,

\[ P(b_{i,1}) = P(b_{i,1}|B_i)P(B_i) + P(b_{i,1}|B_i^c)P(B_i^c) = \frac{1}{nk} + 0(1 - \frac{1}{k}) = \frac{1}{kn} \]

which is intuitive because \( kn \) is the total number of balls.

We are interested in finding the winning ball using two strategies.

**STRATEGY 1**
You are allowed to pick the balls one at a time from any bin you like until you find the winning one. Once a ball is picked, it is removed from the game.

(a) What is \( P(B_j|b_{i,1}) \) in English? Use Bayes’ rule to find \( P(B_j|b_{i,1}) \). The answer should be different for \( j = i \) and \( j \neq i \). Can you interpret the result?

**Solution:** \( P(B_j|b_{i,1}) \) is the probability that bin \( j \) has the winning ball given that on the first trial a wrong ball was picked from bin \( i \). This can be computed as follows:

\[
P(B_j|b_{i,1}) = \frac{P(b_{i,1}|B_j)P(B_j)}{P(b_{i,1})} = \frac{P(b_{i,1}|B_j)P(B_i) + P(b_{i,1}|B_i^c)P(B_i^c)}{P(b_{i,1})}
\]

The denominator is equal to \( (1 - 1/n)1/k + 1(1 - 1/k) = 1 - 1/(nk) \). The numerator will depend on whether \( i = j \) or not.

Case 1: \( i = j \), we get

\[
\frac{(1 - 1/n)1/k}{1 - 1/(nk)} = \frac{n - 1}{nk - 1}
\]

Case 2: \( i \neq j \), we get

\[
\frac{1(1/k)}{1 - 1/(nk)} = \frac{n}{nk - 1}
\]

So any bin other than bin \( i \) has a higher chance of having the winning ball.

(b) This is a general equality that we have seen in class: \( P(X,Y|Z) = P(X|Z)P(Y|X,Z) \). Prove this equality (just to make sure you understand why).

**Solution:**

\[
P(X,Y|Z) = \frac{P(X,Y,Z)}{P(Z)} = \frac{P(Z)P(X|Z)P(Y|X,Z)}{P(Z)} = P(X|Z)P(Y|X,Z)
\]

(c) Find \( P(B_j, b_{j,2}|b_{i,1}) \), for \( j = i \) and \( j \neq i \). This is the probability of selecting the winning ball from bin \( j \) after the removal of a wrong ball from bin \( i \). Use part (b) to compute this probability. Based on your result, suggest a search algorithm for the winning ball.

**Solution:**

\[
P(B_j, b_{j,2}|b_{i,1}) = P(B_j|b_{i,1})P(b_{j,2}|B_j, b_{i,1})
\]

Case 1: \( i = j \), we get

\[
\frac{n - 1}{nk - 1} \frac{1}{n - 1} = \frac{1}{nk - 1}
\]
Case 2: $i \neq j$, we get
\[
\frac{n}{nk - 1} \frac{1}{n} = \frac{1}{nk - 1}
\]
Therefore, all bins have equal chance. Any order is good.

**STRATEGY 2**

You are allowed to pick the balls one at a time from any bin you like until you find the winning one. If a wrong ball is picked, it is returned to the same bin.

Repeat the work for this strategy. *Hint:* only part (c) above should change.

PS: You might want to try playing with small examples first, e.g. when $n = 2$ and $k = 2$, to get a feel of what is going on.

**Solution:**

\[
P(B_j, b_{j,2}|b_{i,1}) = P(B_j|b_{i,1})P(b_{j,2}|B_j, b_{i,1})
\]

Only $P(b_{j,2}|B_j, b_{i,1})$ is affected. Since we now return the ball to its bin:

Case 1: $i = j$, we get
\[
\frac{n - 1}{nk - 1} \frac{1}{n} = \frac{n - 1}{n} \frac{1}{nk - 1} < \frac{1}{nk - 1}
\]

Case 2: $i \neq j$, we get
\[
\frac{n}{nk - 1} \frac{1}{n} = \frac{1}{nk - 1}
\]

Therefore, we are more likely to find the winning ball on our second trial if we pick a bin other than bin $i$. In fact, a round robin strategy will be the best in this case.