Problem 0: Readings
Read note 6 on the course web page.

Problem 1: Mixture
Suppose that your prior for \( \mu \) is \( \frac{2}{3} : \frac{1}{3} \) mixture of \( N(0, 1) \) and \( N(1, 1) \), and that a single observation is made \( x \sim N(\mu, 1) \) and turns out to be equal to 2. What is the posterior probability that \( \mu > 1 \).

Solution: Based on note 6, the posterior for each of the above priors when considered separately, is given by (when \( x = 2 \) as given):

\[ \mu | x \sim N\left( \frac{\sigma^2 \beta + \tau^2 x}{\sigma^2 + \tau^2}, \frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2} \right) \]

\( \mu | x \sim N(1, 1/2) \) given first prior \( g(\mu) \)

\( \mu | x \sim N(3/2, 1/2) \) given second prior \( h(\mu) \)

Therefore \( f(\mu|x) \) is a mixture of the above two posteriors, with mixing factors \( \alpha(2) \) and \( \beta(2) = 1 - \alpha(2) \).

\[ f(\mu|x = 2) = \alpha(2)N(1, 1/2) + \beta(2)N(3/2, 1/2) \]

where

\[ \frac{\alpha(x)}{1 - \alpha(x)} = \frac{\alpha \int f(x|\mu)g(\mu)d\mu}{\beta \int f(x|\mu)h(\mu)d\mu} \]

This leads to:

\[ \alpha(x) = \frac{2 \int f(x|\mu)g(\mu)d\mu}{\int f(x|\mu)g(\mu)d\mu + \int f(x|\mu)h(\mu)d\mu} \]

\[ f(x|\mu)g(\mu) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(\mu-x)^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{1}{2}} e^{-\frac{x^2}{2}} \]

\[ = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{1}{2}} e^{-\frac{(\mu-x)^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-x^2/4} \]

\[ = \frac{1}{\sqrt{2\pi}} e^{-\frac{(\mu-x)^2}{2(\sqrt{1/2})^2}} = \frac{1}{\sqrt{2\pi}} e^{-x^2/4} \]

\[ = \frac{1}{\sqrt{2\pi}} e^{-\frac{(\mu-x)^2}{2(\sqrt{1/2})^2}} = \frac{1}{\sqrt{2\pi}} e^{-x^2/4} \]
Therefore,
\[ \int f(x|\mu) g(\mu) d\mu = \frac{\sqrt{1/2}}{\sqrt{2\pi}} e^{-x^2/4} \]
Similarly, we get:
\[ f(x|\mu) h(\mu) = \frac{\sqrt{1/2}}{\sqrt{2\pi}} e^{-(x-1)^2/4} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(\mu - (x+1/2))^2}{2(\sqrt{1/2})^2}} \]
Therefore,
\[ \int f(x|\mu) h(\mu) d\mu = \frac{\sqrt{1/2}}{\sqrt{2\pi}} e^{-(x-1)^2/4} \]
Setting \( x = 2 \), we get \( \alpha(2) \approx 0.49 \). Therefore,
\[ f(\mu|x) = 0.49 \frac{1}{\sqrt{2\pi}\sqrt{0.5}} e^{-\frac{\mu - 1.5^2}{2(0.5)^2}} + 0.51 \frac{1}{\sqrt{2\pi}} e^{-\frac{\mu - 1.5^2}{2(0.5)^2}} \]
We need to integrate the above from 1 to \( \infty \) to obtain the posterior probability that \( \mu > 1 \). The first part should evaluate to 0.9 \( \times \) 0.5 because it consists of a normal density with mean 1. For the second, we have to translate it into a standard normal to figure it out. So we will be looking at \( z = (1 - 1.5) / \sqrt{0.5} = -0.707 \). We need \( 1 - \Phi(-0.707) = 0.76 \). Therefore, we get:
\[ 0.49 \times 0.5 + 0.51 \times 0.76 \approx 0.634 \]
Problem 2:
Assume \( X_i \sim_{iid} N(\mu, \sigma^2) \) and \( \mu \) is believed to be \( \mu_0 \) with probability 1/2 and \( \mu \sim N(\mu_0, \sigma^2) \) with probability 1/2.

A number of observations, \( n \), reveal \( \bar{x} = \mu_0 + 1.96\sigma / \sqrt{n} \). What is the probability that \( \mu = \mu_0 \) when \( n = 5 \) and \( n = 50 \)? Comment on the findings.

Solution:
\[ P(\mu = \mu_0|\bar{x}) = \frac{f(\bar{x}|\mu = \mu_0) \times 0.5}{f(\bar{x}|\mu = \mu_0) \times 0.5 + \int f(\bar{x}|\mu)N(\mu_0, \sigma^2) d\mu \times 0.5} \]
So
\[ P(\mu = \mu_0|\bar{x}) = \frac{e^{-1.9208}}{e^{-1.9208} + \int e^{-\frac{(\bar{x}-\mu)^2}{2\sigma^2}} N(\mu_0, \sigma^2) d\mu} \]
Now let’s rearrange the expression inside the integral:
\[ e^{-\frac{\mu^2}{2\sigma^2}/n} e^{-\frac{(\bar{x}-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{n\bar{x}^2+n\mu^2-2\mu n\bar{x}+\mu^2\bar{x}^2-2\mu n\bar{x}}{2\sigma^2}} \]
\[ = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(n+1)\mu^2-2\mu(n+\mu_0)+n\bar{x}^2+n\mu_0^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\mu^2-2\mu(n+\mu_0)+n\bar{x}^2+n\mu_0^2}{2\sigma^2/(n+1)}} \]
\[ = \frac{1}{\sqrt{2\pi}\sigma} \sqrt{n+1} e^{-\frac{(n+1)\mu^2-2\mu(n+\mu_0)+n\bar{x}^2+n\mu_0^2}{2\sigma^2/(n+1)}} \]
So when we integrate we get:
\[ \frac{1}{\sqrt{n+1}} e^{-\frac{n\bar{x}^2+n\mu_0^2}{2\sigma^2/(n+1)}} = \frac{1}{\sqrt{n+1}} e^{-\frac{(\bar{x}-\mu)^2}{2\sigma^2/(n+1)}} = \frac{1}{\sqrt{n+1}} e^{-1.9208} \]
For any value of \( n \), we have:

\[
P(\mu = \mu_0 | \bar{x}) = \frac{e^{-1.9208}}{e^{-1.9208} + \frac{1}{\sqrt{n+1}} e^{\frac{1}{2n+1}}}
\]

When \( n = 5 \) this is approximately 0.33, and when \( n = 50 \) this is approximately 0.52. This is essentially Lindley’s paradox. The larger \( n \) is, the more likely that \( \mu = \mu_0 \), even though a z-score for \( \bar{x} \) corresponds to \( \Phi(-1.96) \) which is 0.025.

**Problem 3**

A hypothesis may be strongly rejected by a test of significance and yet be awarded high odds by a Bayesian analysis based on a mixture of a small prior probability of that hypothesis, and a diffuse density of the remaining probability. This is known as Lindley’s paradox.

In this problem you are asked to exhibit this paradox using only one sample point \( x \), where \( X \sim N(\mu, \sigma^2) \), and a mixture of the following prior densities: \( \mu = \mu_0 \) with probability \( p \), and the rest of the probability is distributed according to a normal density \( \mu \sim N(\beta, \tau^2) \), such that \( \sigma/\tau \rightarrow 0 \) (making it diffuse).

In other words, show that \( P(\mu = \mu_0 | x) \) goes to 1 when \( \sigma/\tau \rightarrow 0 \) for some choice of \( x \) where \( \Phi(-|x-\mu_0|/\sigma) \approx 0 \).

**Solution:**

\[
\frac{\alpha(x)}{1 - \alpha(x)} = \frac{p \int_{-\infty}^{\infty} f(x|\mu) \delta(\mu - \mu_0) \, d\mu}{(1 - p) \int_{-\infty}^{\infty} f(x|\mu) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\mu - \mu_0)^2}{2\sigma^2}} \, d\mu}
\]

Rearranging terms of the integral in the denominator, we get:

\[
\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\mu - \mu_0)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(\mu - \mu_0)^2}{2\sigma_0^2}} = \frac{1}{2\pi\sigma} e^{-\frac{(\mu - \mu_0)^2}{2\sigma_0^2} - \frac{\tau^2 + \sigma^2 \beta^2}{2\tau^2 + \sigma^2}}
\]

\[
= \frac{1}{\sqrt{\tau^2 + \sigma^2}} \frac{e^{-\frac{(\mu - \mu_0)^2}{\tau^2 + \sigma^2}}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(\mu - \mu_0)^2}{2\tau^2 + \sigma^2}}
\]

Therefore, when we integrate we get:

\[
\frac{1}{\sqrt{\tau^2 + \sigma^2}} \frac{e^{-\frac{(\mu - \mu_0)^2}{\tau^2 + \sigma^2}}}{\sqrt{2\pi}}
\]

Letting \( \sigma/\tau \rightarrow 0 \), we get:

\[
\frac{1}{\sqrt{\tau^2}}
\]

Now

\[
f(x|\mu_0) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu_0)^2}{2\sigma^2}}
\]

For any value of \( x = \mu_0 \pm c\sigma \), the above is a constant multiplied by \( \frac{1}{\sqrt{2\pi}\sigma} \).
Therefore,

\[ \frac{\alpha(x)}{1 - \alpha(x)} \propto \frac{\tau}{\sigma} \]

which goes to infinity as \( \sigma/\tau \to 0 \). Therefore, \( \alpha(x) \to 1 \).