MARKOV CHAINS.

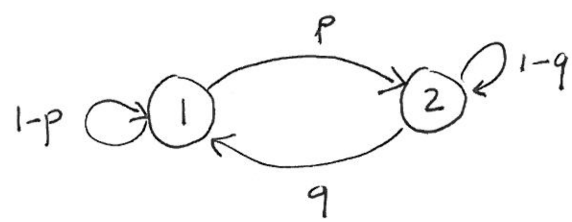
CONSIDER A SET OF STATES $S = \{1, 2, \dots, k\}$ AND ASSUME STATES EVOLVE OVER TIME IN DISCRETE TIME STEPS. LET X_n BE THE STATE AT TIME n. WE HAVE A MARKOV CHAIN IF $(m < n)$:

$$P(X_n = j \,/\, X_0, X_1, X_2, \dots, X_m = i) = P(X_n = j \,/\, X_m = i)$$

Further more WHEN $m = n-1$, THIS PROBABILITY IS GIVEN BY P_{ij}, WHERE $\underline{P}$ is THE "TRANSITION" MATRIX, AND $\sum_{j=1}^{K} P_{ij} = 1$.

A SIMPLE IS GIVEN BY THE FOLLOWING MATRIX AND ALSO ILLUSTRATED BY A STATE DIAGRAM.

$$\underline{P} = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$$



AS A MOTIVATION TO STATE AN INTERESTING PROPERTY CONSIDER $P(X_2 = 2 \,|\, X_0 = 1)$. WE CAN REWRITE THIS AS:

$$P(X_2 = 2 / X_0 = 1) = \sum_i P(X_2 = 2 \,/\, X_1 = i, X_0 = 1)\, P(X_1 = i / X_0 = 1)$$

$$= \sum_i P(X_2 = 2 / X_1 = i)\, P(X_1 = i / X_0 = 1)$$

$$= P(X_2 = 2 \,/\, X_1 = 1)\, P(X_1 = 1 / X_0 = 1) + P(X_2 = 2 / X_1 = 2)\, P(X_1 = 2 / X_0 = 1)$$

$$= p(1-p) + (1-q)\,p$$

WE CAN OBSERVE THAT THIS IS THE ENTRY IN THE FIRST ROW AND SECOND COLUMN OF P^2.

IN GENERAL,

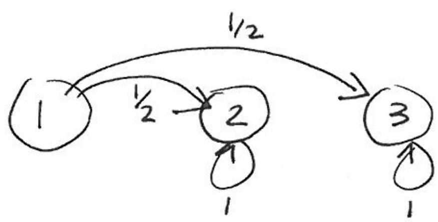
$$P(X_n = j \mid X_m = i) = P_{ij}^{\,n-m} \quad \text{FOR } n > m.$$

TWO PROPERTIES

THE MATRIX P IS <u>IRREDUCIBLE</u> IFF

$$\forall\, i,j \quad P(X_n = j \mid X_0 = i) > 0 \text{ FOR SOME } n.$$

IN TERMS OF THE STATE DIAGRAM, THIS MEANS THERE IS A PATH FROM i TO j (NOT NECESSARILY AN IMMEDIATE TRANSITION). THE PREVIOUS MARKOV CHAIN IS IRREDUCIBLE. THIS ONE (BELOW) IS NOT.



WHAT IS IMPORTANT ABOUT THE IRREDUCIBILITY? IT DEFINES A UNIQUE STATIONARY DISTRIBUTION OF STATES. MORE PRECISELY, IF CHAIN IS IRREDUCIBLE THEN THERE EXISTS A UNIQUE PROBABILITY VECTOR $\pi > 0$ $(\sum \pi_i = 1)$ SUCH THAT

$$\pi P = \pi.$$

WHY DO WE CALL π A STATIONARY DISTRIBUTION?

LET'S EXAMINE THIS FOR A GIVEN i.

$$\begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_k \end{bmatrix} \begin{bmatrix} & P_{1i} & \\ & P_{2i} & \\ & \vdots & \\ & P_{ki} & \end{bmatrix} = \begin{bmatrix} \cdots & \pi_i & \cdots \end{bmatrix}$$

WE HAVE $\pi_1 P_{1i} + \pi_2 P_{2i} + \cdots + \pi_k P_{ki} = \pi_i$

IF π is A PROBABILITY DISTRIBUTION OVER STATES IN TIME n, THE LEFT HAND SIDE IS THE PROBABILITY OF BEING IN STATE i AT TIME $n+1$. IF THIS IS EQUAL TO π_i, THEN THE PROB. OF STATE i HAS NOT CHANGED. THIS IS TRUE FOR ALL i.

* $P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$ THIS IS IRREDUCIBLE IF $p, q > 0$

$\pi = \begin{bmatrix} \frac{q}{p+q} & \frac{p}{p+q} \end{bmatrix}$ IS STATIONARY (AND UNIQUE). VERIFY

* $P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ THIS IS NOT IRREDUCIBLE

ANY VECTOR OF THE FORM

$[0 \ \ p \ \ 1-p]$ IS STATIONARY.

(NOT UNIQUE)

HOW DO WE FIND THE STATIONARY DISTRIBUTION?

WE CAN SOLVE THE SYSTEM OF EQUATIONS GIVEN BY

$$\pi P = \pi$$

USING $\sum_i \pi_i = 1$ AS AN ADDITIONAL EQUATION.

$$[\pi_1 \ \pi_2 \ \cdots \ \pi_k] \underbrace{\begin{bmatrix} & & 1 \\ & & 1 \\ & & \vdots \\ & & 1 \end{bmatrix}}_{P} = [\pi_1 \ \pi_2 \ \cdots \ \pi_{k-1} \ 1]$$

SO LET $Q = P - I$ WITH LAST COLUMN REPLACED BY ALL ONES. THEN

$$\pi Q = [0 \ 0 \ \cdots \ 0 \ 1]$$

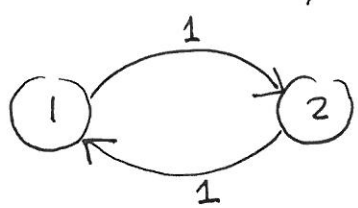
$$\pi = [0 \ 0 \ \cdots \ 0 \ 1] \, Q^{-1}.$$

A MATRIX P IS APERIODIC IF FOR EVERY i

$$\mathrm{GCD} \{ n : P(X_n = i \mid X_0 = i) > 0 \} = 1$$

IN OTHER WORDS, IF WE CONSIDER THE LENGTHS OF ALL CYCLES THAT BRINGS US BACK TO i, THOSE LENGTHS HAVE NO COMMON DIVISOR (EXCEPT 1).

THIS CHAIN IS IRREDUCIBLE, BUT NOT APERIODIC



MAIN RESULT

IF A MARKOV CHAIN IS IRREDUCIBLE, THEN

$$\frac{1}{n+1}\sum_{t=0}^{n} f(X_t) \xrightarrow[n\to\infty]{} \sum_i \pi_i f(i)$$

WHERE π IS THE STATIONARY DISTRIBUTION, FOR EVERY BOUNDED FUNCTION f.

(THIS IS ANALOGOUS TO THE STRONG LAW OF LARGE NUMBERS WHERE $X_0, X_1, \ldots$ ARE INDEPENDENT. HERE THEY ARE NOT).

IN ADDITION, IF CHAIN IS APERIODIC, THEN

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \pi_1 & \cdots & \pi_k \\ \pi_1 & \cdots & \pi_k \\ & \vdots & \\ \pi_1 & \cdots & \pi_k \end{bmatrix}$$

i.e.

$$\lim_{n\to\infty} P(X_n = i / X_0) = \pi_i$$

IN THIS CASE $\sum_i \pi_i f(i) \equiv E[f(X)]$

REVERSIBILITY

CONSIDER THE REVERSE PROCESS GIVEN BY

$$P\left(X_n = i \,/\, X_{n+1} = j,\, X_{n+2} = k,\, X_{n+3} = \cdots\right)$$

$$= \frac{P(X_n = i, X_{n+1} = j, X_{n+2} = k, X_{n+3} = \ldots)}{P(X_{n+1} = j, X_{n+2} = k, X_{n+3} = \ldots)}$$

$$= \frac{P(X_n = i)\, P_{ij}\, P_{jk} \cdots\cdots}{P(X_{n+1} = j)\, P_{jk} \cdots}$$

IF STATIONARY, THIS IS $\dfrac{\pi_i P_{ij}}{\pi_j}$

So $P(X_n = i \mid X_{n+1} = j, X_{n+2}, X_{n+3}, \dots)$

$$= P(X_n = i \mid X_{n+1} = j)$$

SO THE REVERSE PROCESS ALSO EXHIBITS THE MARKOV PROPERTY DEPENDENCE OF THE MOST RECENT STATE.

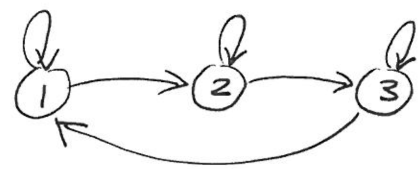
WE CALL A MARKOV CHAIN REVERSIBLE IF

$$P(X_n = i \mid X_{n+1} = j) = P(X_{n+1} = i \mid X_n = j)$$

(REVERSE TIME)

SO $\quad \frac{\pi_i}{\pi_j} P_{ij} = P_{ji} \Rightarrow \boxed{\pi_i P_{ij} = \pi_j P_{ji}}$

NOT ALL MARKOV CHAINS ARE REVERSIBLE. HERE'S ONE THAT IS NOT.



IF ALL TRANSITION PROBABILITIES ARE POSITIVE, THIS IS IRREDUCIBLE AND APERIODIC. BUT IF $P_{ij} > 0$, THEN $P_{ji} = 0$. SO IT CANNOT SATISFY THE REVERSIBILITY CONDITION.

WHY IS THIS IMPORTANT? IF WE FIND π SUCH THAT

$$\pi_i P_{ij} = \pi_j P_{ji} \quad \forall\, i,j \quad \text{THEN}$$

π IS STATIONARY.

$$\sum_i \pi_i P_{ij} = \sum_i \pi_j P_{ji} = \pi_j \sum_i P_{ji} = \pi_j$$

So $\pi P = \pi$.