Computational Biology

Lecture 17

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Diagram of Reality and Desire

- α is the reality
- β is what is desired

- The diagram $RD_{\beta}(\alpha)$ is the diagram of reality and desire:

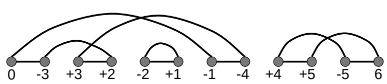

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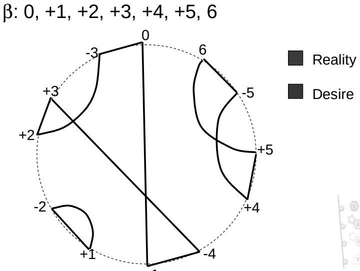
Diagram of Reality and Desire

(for better visualization)

- Extended α: 0, +3, –2, –1, +4, –5, 6
- Extended β: 0, +1, +2, +3, +4, +5, 6



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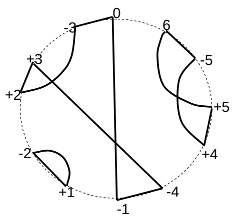
Definition

Two reality edges on the same cycle *converge* if they are traversed in the same direction (either clockwise or counterclockwise) on the cycle.



Example:

(+3,+2) and (-1,-4) converge

(0,-3) and (+3,+2) diverge

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Reversals and $c(\alpha)$

Let ρ be a reversal defined by two reality edges *e* and *f*, then:

– If *e* and *f* belong to different cycles, $c(\alpha\rho) = c(\alpha) - 1$

– If *e* and *f* belong to the same cycle and converge, $c(\alpha\rho) = c(\alpha)$

– If *e* and *f* belong to the same cycle and diverge, $c(\alpha\rho) = c(\alpha) + 1$

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Another lower bound

- Let $\rho_1, \ldots, \rho_t$ be such that $\alpha\rho_1 \ldots \rho_t = \beta$, then:

$$\left.\begin{array}{l} c(\alpha\rho_1) - c(\alpha) \leq 1 \\ c(\alpha\rho_1\rho_2) - c(\alpha\rho_1) \leq 1 \\ \ldots \\ c(\alpha\rho_1 \ldots \rho_t) - c(\alpha\rho_1 \ldots \rho_{t-1}) \leq 1 \end{array}\right| \Rightarrow c(\alpha\rho_1 \ldots \rho_t) - c(\alpha) = n + 1 - c(\alpha) \leq t$$

- Let $t = d(\alpha)$, then $\boxed{\boldsymbol{d(\alpha) \geq n + 1 - c(\alpha)}}$

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Better lower bound

- The lower bound $n + 1 - c(\alpha)$ is better than $b(\alpha)/2$.
- For most signed permutations, it comes very close to the actual reversal distance.
- It does not always work. We cannot always choose two divergent edges (doing so will increase the number of cycles by 1 each time and achieve the exact bound).
- To understand why, we need to define some concepts.

Good / Bad cycles

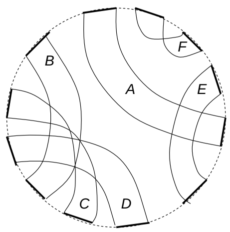
The easy stuff first….

- A cycle is **good** iff it has two reality edges that diverge
- A cycle is **bad** iff all its reality edges converge
- A cycle is **proper** iff is contains more than two edges (we will only look at these).

Interleaving

- Assume the special way of drawing $RD(\alpha)$
  - either the counterclockwise circle where reality edges are on the circumference and desire edges are chords
  - or reality edges are on the line and desire edges on top
- Two cycles interleave iff a desire edge of one crosses a desire edge of the other
- In general we can define this using intervals without assuming a special way of drawing $RD(\alpha)$

Interleaving



A and *E* interleave

B and *C* interleave

C and *D* interleave

B and *D* interleave

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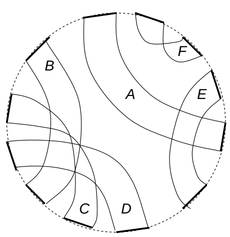
Components



We have three components:

{F}

{A, E}

{B, C, D}

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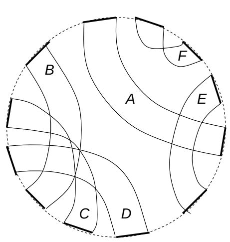
Good / Bad components



We have three components:

{F} : good

{A, E} : bad

{B, C, D} : good

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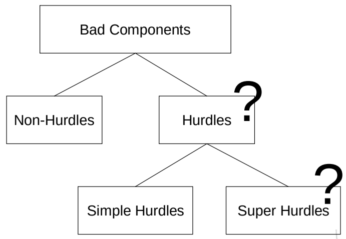
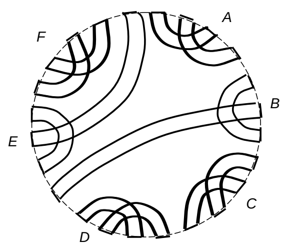
Classification of bad components

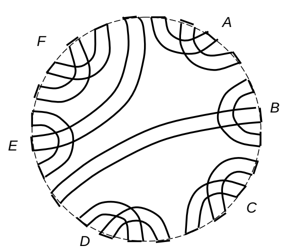


Hurdles

A bad component is a hurdle iff it does not separate between any two components



Super Hurdles

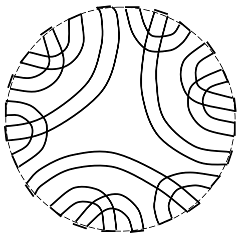


F is a super hurdle, it protects *E*

All other hurdles are Simple hurdles

Fortress

A signed permutation α is called a fortress iff $RD(\alpha)$ has an odd number of hurdles and all of them are super hurdles



The smallest fortress, with 3 super hurdles

New lower bound

$d(\alpha) \geq n + 1 - c(\alpha) + h(\alpha)$

Any reversal ρ acts on two reality edges, and can "destroy" at most two hurdles (by definition, hurdles cannot separate between hurdles), therefore

$\Delta h = h(\alpha\rho) - h(\alpha) \geq -2$

three cases:

– $\Delta c = 1$, then ρ acts on a good cycle and $\Delta h = 0$ (no hurdles destroyed), $\Delta(c - h) = 1$

– $\Delta c = 0$, then ρ acts on a bad cycle and $\Delta h \geq -1$ (at most one hurdle destroyed), $\Delta(c - h) \leq 1$

– $\Delta c = -1$, then $\Delta h \geq -2$ anyway (at most two hurdles destroyed), $\Delta(c - h) \leq 1$

Exact traversal distance

$d(\alpha) = n + 1 - c(\alpha) + h(\alpha)$, α non-fortress

$d(\alpha) = n + 1 - c(\alpha) + h(\alpha) + 1$, α fortress

where $h(\alpha)$ is the number of hurdles

Approach

- We will achieve the lower bound $n + 1 - c(\alpha) + h(\alpha)$ for a non-fortress permutation α
- Each time we will find a reversal ρ such that $c(\alpha\rho) - h(\alpha\rho) = c(\alpha) - h(\alpha) + 1$
- Call such a reversal a safe reversal
- $c(\alpha) - h(\alpha)$ can be at most n+1 (when $\alpha = \beta$); therefore, we will perform only n+1 – $c(\alpha) + h(\alpha)$ reversals

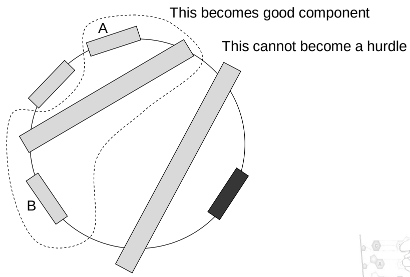
Safe reversal Kind I

- A reversal ρ defined by two divergent reality edges of a good cycle that does not lead to the creation of bad components
- Safe: increases c by 1, keeps h unchanged; therefore, $c(\alpha\rho) - h(\alpha\rho) = c(\alpha) - h(\alpha) + 1$
- Fact: if we have a good component, then there exists a safe reversal of kind I
  (we are not going to prove this)

Safe reversal Kind II (hurdle merging)

- Define two opposite hurdles *A* and *B* such that the number of hurdles between *A* and *B* is the same on either sides of the circle.
- A reversal ρ defined by two reality edges of two opposite hurdles (i.e. the number of hurdles must be even)
- ρ destroys two hurdles which become a good component with any non-hurdle that separates them.
- ρ does not create new hurdles.
- Safe: decreases c by 1, decreases h by 2; therefore, $c(\alpha\rho) - h(\alpha\rho) = c(\alpha) - h(\alpha) + 1$

Illustration



Safe reversal Kind III (hurdle cutting)

- A reversal ρ defined by two convergent reality edges of a bad cycle in a simple hurdle when number of hurdles is odd.
- ρ destroys the hurdle (makes it good component), and does not create new hurdles (otherwise it is a super hurdle), and results in even number of hurdles.
- Safe: keeps *c* unchanged, decreases *h* by 1; therefore, $c(\alpha\rho) - h(\alpha\rho) = c(\alpha) - h(\alpha) + 1$

Non-fortress

- If α is not a fortress, then we can always find a safe reversal
- Proof:
  - If there is a good component, then there is a kind I safe reversal.
  - If there are no good components and the number of hurdles is even, then there is a Kind II safe reversal (hurdle merging)
  - If there are no good components and the number of hurdles is odd, then there is a simple hurdle (otherwise α is a fortress), and there is a Kind III safe reversal (hurdle cutting). After that point the number of hurdles is always even.

Algorithm

given distinct α and β

repeat
  if there is a good component in $RD_{\beta}(\alpha)$
  then pick a Kind I reversal

  else if $h(\alpha)$ is even
    then pick a Kind II reversal (merging two opposite hurdles)

  else if $h(\alpha)$ is odd
    then pick a Kind III reversal (cutting simple hurdle)

until $\alpha = \beta$

What if α is a fortress

- Then we have an odd number of hurdles non of which is simple
  - We cannot cut a super hurdle since this will create a new hurdle
  - We can merge two super hurdles (not opposite), but there is a danger of creating a new hurdle!
- FACT: merging two super hurdles in a fortress with $h \neq 3$ super hurdles results in a fortress with $h - 2$ super hurdles.
- Therefore, we create a new hurdle only when we have a forest with 3 super hurdles resulting is a distance of $n + 1 - c(\alpha) + h(\alpha) + 1$ for a fortress permutation.
- This is optimal for a fortress because any fortress has to have one unsafe reversal, but we will not prove it here.

Algorithm

given distinct α and β

repeat
  if there is a good component in $RD_{\beta}(\alpha)$
  then pick a Kind I reversal

  else if $h(\alpha)$ is even
    then pick a Kind II reversal (merging two opposite hurdles)

  else if $h(\alpha)$ is odd and there is a simple hurdle
    then pick a Kind III reversal (cutting simple hurdle)

  else //fortress
    merge any two hurdles (this will result in a fortress if $h(\alpha) \neq 3$)

until $\alpha = \beta$

Running time (initialization of structure)

- Constructing $RD(\alpha)$ takes $O(n)$ time (determine reality and desire edges)
- Finding cycles can be done in $O(n)$ time
- For each cycle we have to determine whether it is good or bad, this takes $O(n)$ time for each cycle for a total of $O(n^2)$ time
- Interleaving of cycles can be done in $O(n^2)$ time by examining every pair of desire edges
- Determining good and bad components, non-hurdles, simple hurdles, and super hurdles can be done in $O(n)$ time
- So far $O(n^2)$ time

Running time (cont.)

- The most time consuming part is identifying whether a safe reversal of Kind I exists
- Since a reversal is defined by two edges, we have $O(n^2)$ reversals to try
- For each reversal we have to see whether a bad component will be created (this can be done in $O(n^2)$ time as discussed above by computing RD($\alpha\rho$))
- Therefore, we spend $O(n^4)$ time
- This is performed at most n times yielding an $O(n^5)$ time algorithm