Boolean function

$$f : \{0,1\}^n \longrightarrow \{0,1\} \qquad \{0,1\}^n = \underbrace{\{0,1\} \times \{0,1\} \times \dots \times \{0,1\}}_{n \text{ times}}$$

Example: $f : \{0,1\}^3 \longrightarrow \{0,1\}$

x	y	z	$f(x,y,z)$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Any Boolean function can be constructed using $\{\neg, \vee, \wedge\}$ operators

We say $\{\neg, \vee, \wedge\}$ is UNIVERSAL

$$f(x,y,z) = (\neg x \wedge \neg y \wedge z) \vee (\neg x \wedge y \wedge \neg z) \vee (x \wedge \neg y \wedge \neg z) \vee (x \wedge y \wedge z)$$

x	y	z	$f(x,y,z)$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

x	y	z
0	0	1

$\neg x \wedge \neg y \wedge z$

0	1	0

$\neg x \wedge y \wedge \neg z$

1	0	0

$x \wedge \neg y \wedge \neg z$

1	1	1

$x \wedge y \wedge z$

$\vee$

$\{\neg, \wedge\}$ is universal

$\{\neg, \vee\}$ is universal

De Morgan's Law:

$$\left.\begin{aligned} \neg(A \wedge B) &= \neg A \vee \neg B \\ \neg(A \vee B) &= \neg A \wedge \neg B \end{aligned}\right\} \text{proof? Truth Table}$$

Example: $\overbrace{a \notin \mathbb{Q}}^{A} \vee \overbrace{b \notin \mathbb{Q}}^{B}$

Negate above statement: $a \in \mathbb{Q} \wedge b \in \mathbb{Q}$

Commutative:

$$A \wedge B = B \wedge A$$

$$A \vee B = B \vee A$$

Associative:

$$A \wedge (B \wedge C) = (A \wedge B) \wedge C = A \wedge B \wedge C$$

$$A \vee (B \vee C) = (A \vee B) \vee C = A \vee B \vee C$$

Distributive:

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$$

Properties of Implication

P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

1) P is true and $(P \Rightarrow Q)$ is true, then Q is true.
(look at last row) [direct proof]

2) $(\neg Q \Rightarrow \neg P) = (P \Rightarrow Q)$ [proof by contrapositive]

3) $(\neg P \Rightarrow \text{False})$ is true, then P is true

[proof by contradiction]

(look at row in which Q is False but $P \Rightarrow Q$ is true) (1st row)

4) $(P \Rightarrow Q)$ is true and $(Q \Rightarrow R)$ is true

then $(P \Rightarrow R)$ is true. [transitive]

Let's prove property (4):

by case analysis of P.

P is False : $(P \Rightarrow R)$ is true regardless of R

P is True : P is true and $(P \Rightarrow Q)$ is true

then Q is true.

Q is true and $(Q \Rightarrow R)$ is true, then

R is true.

Therefore $(P \Rightarrow R)$ is true

Example 1 : Prove $\underbrace{n \text{ odd}}_{P} \Rightarrow \underbrace{n^2 \text{ is odd}}_{Q}$ (is true)

Definition:

n odd : $n = 2 \cdot k + 1 \; , \; k \in \mathbb{Z}$

n even : $n = 2 \cdot k \; , \; k \in \mathbb{Z}$

Direct proof

P: n is odd $\xRightarrow[\text{def.}]{} n = 2k + 1 \; , \; k \in \mathbb{Z}$

$$n = 2k+1 \xRightarrow[\text{algebra}]{} n^2 = (2k+1)^2 = 4k^2 + 4k + 1$$

$$= 2(\underbrace{2k^2 + 2k}_{k' \in \mathbb{Z}}) + 1$$

$n^2 = 2k' + 1 \xRightarrow[\text{def.}]{} n^2$ is odd.

We also proved: n^2 is even $\Rightarrow$ n is even.

Example 2: Prove $\underbrace{ab \notin \mathbb{Q}}_{A} \implies \underbrace{(a \notin \mathbb{Q} \vee b \notin \mathbb{Q})}_{B}$

Consider the contrapositive:

$$\neg B \implies \neg A$$

contrapositive

$$a \in \mathbb{Q} \wedge b \in \mathbb{Q} \implies ab \in \mathbb{Q}$$

- $a \in \mathbb{Q} \implies a = \dfrac{x}{y}, \; x \in \mathbb{Z}$ and $y \in \mathbb{N}$
- $b \in \mathbb{Q} \implies b = \dfrac{z}{w}, \; z \in \mathbb{Z}$ and $w \in \mathbb{N}$

$$a \in \mathbb{Q} \wedge b \in \mathbb{Q} \implies a = \frac{x}{y} \text{ and } b = \frac{z}{w}$$

$$\begin{matrix} P \Rightarrow Q \\ W \Rightarrow R \end{matrix} \vdash P \wedge W \Rightarrow Q \wedge R$$

$$a = \frac{x}{y} \quad \text{and} \quad b = \frac{z}{w} \implies$$

$$ab = \frac{x}{y} \cdot \frac{z}{w} = \frac{x \cdot z}{y \cdot w}, \quad x.z \in \mathbb{Z}, \; y.w \in \mathbb{N}$$

$$\implies ab \in \mathbb{Q}$$

Example: Prove $\sqrt{2}$ is irrational.

by contradiction

$P: \sqrt{2} \notin \mathbb{Q}$

$\neg P: \sqrt{2} \in \mathbb{Q}$

$\sqrt{2} \in \mathbb{Q} \Rightarrow \sqrt{2} = \frac{a}{b}$ (a & b are integers and $\frac{a}{b}$ is irreducible)

$\Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2b^2 \Rightarrow a^2$ is even $\Rightarrow$ a is even.

$$\sqrt{2} \in \mathbb{Q} \Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow b^2 = \frac{a^2}{2} = \frac{2k \times 2k}{2} = \frac{4k^2}{2} = 2k^2$$

$\Rightarrow b^2$ is even $\Rightarrow$ b is even

a is even and b is even $\Rightarrow \frac{a}{b}$ is reducible

$$\sqrt{2} \in \mathbb{Q} \Rightarrow \underbrace{\left(\frac{a}{b} \text{ is reducible and } \frac{a}{b} \text{ is irreducible}\right)}_{\text{False}}$$