Infinity and Beyond

S is Countable set :

- S is Finite , or
- There exists a function $f : S \to \mathbb{N}$ (or $f: \mathbb{N} \to S$) that is a bijection.
  (S and $\mathbb{N}$ have the same Cardinality)

Example: $f : \mathbb{N} \to \{2, 4, 6, 8, \dots\}$

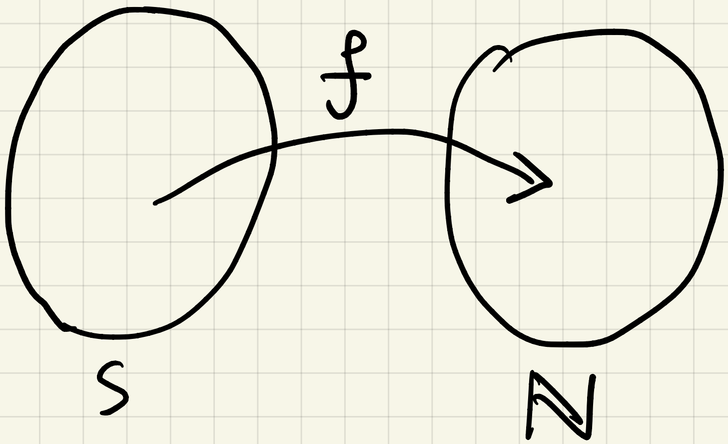
$$f(x) = 2x$$

one-to-one : $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$

onto: $y \in \{2, 4, 6, 8, \dots\}$, let $x = \frac{y}{2} \in \mathbb{N}$, $f(x) = y$

Alternative definition:

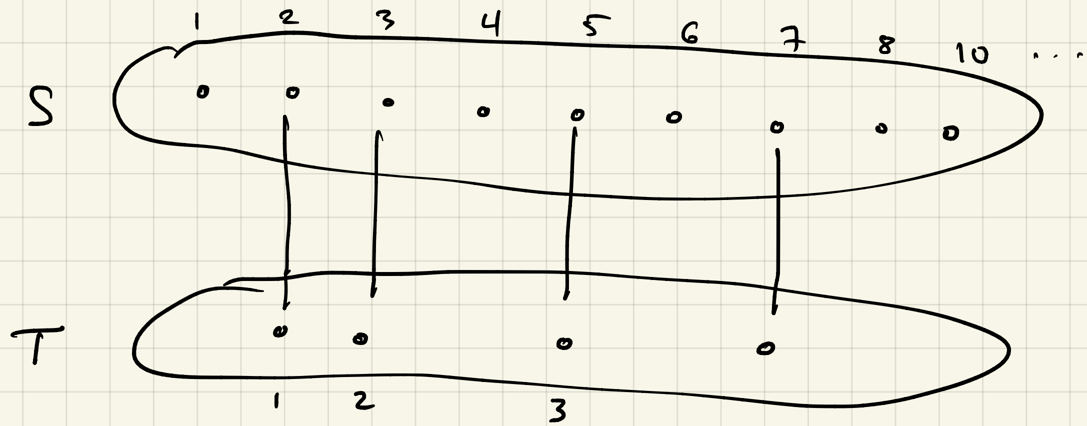
S is countable if there exist a function $f: S \to \mathbb{N}$ that is one-to-one



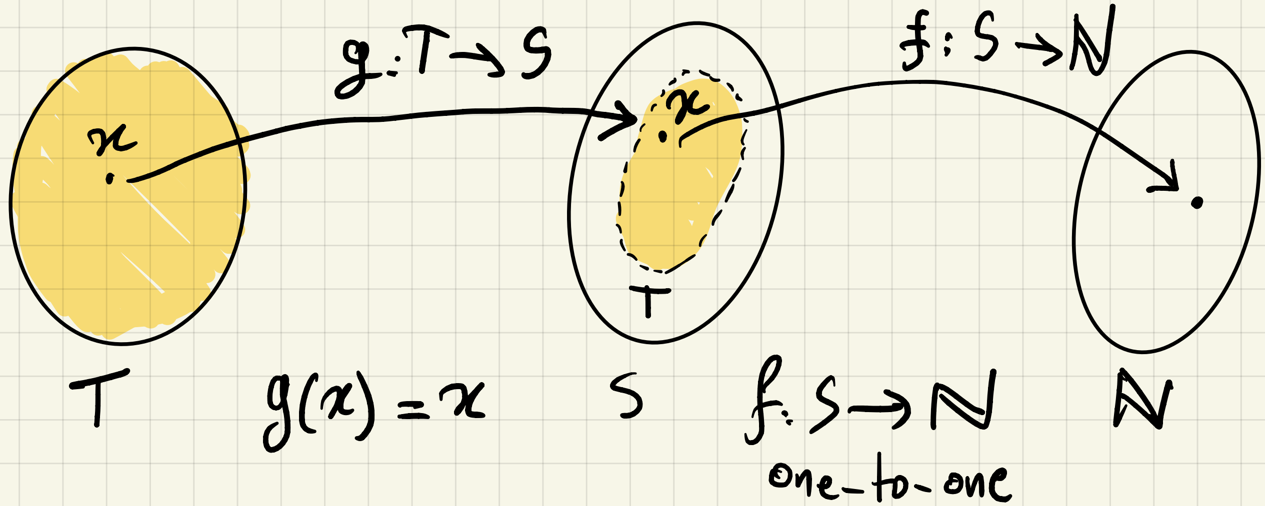
Informal interpretation: f "orders" the elements of S.

Each element must have a finite rank

- If S is countable and $T \subset S$, then T is countable



- Preserve the relative order of elements in T.
- each element in T has a rank that is at most its rank is S.
- each element in T has a finite rank

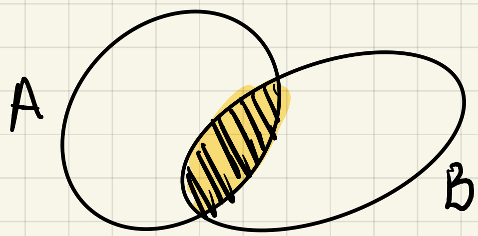


$f \circ g : T \to \mathbb{N}$

$f \circ g(x) = f(g(x))$

is one-to-one

- If A and B are countable, then $A \cap B$ is countable



$(A \cap B) \subset A$

since A is countable, then

$A \cap B$ is also countable.

- If A and B are countable, then $A \cup B$ is countable

$$A = \{ a_1, a_2, a_3, \cdots \}$$

$$B = \{ b_1, b_2, b_3, \cdots \}$$

Consider $A \cup B = \{ a_1, a_2, a_3, \cdots\cdots, b_1, b_2, b_3, \cdots \}$

$$A \cup B = \{ a_1, b_1, a_2, b_2, a_3, b_3, \cdots\cdots \}$$

Rank of b_i : $2i$

a_i : $2i - 1$

Is $\mathbb{Z}$ countable

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

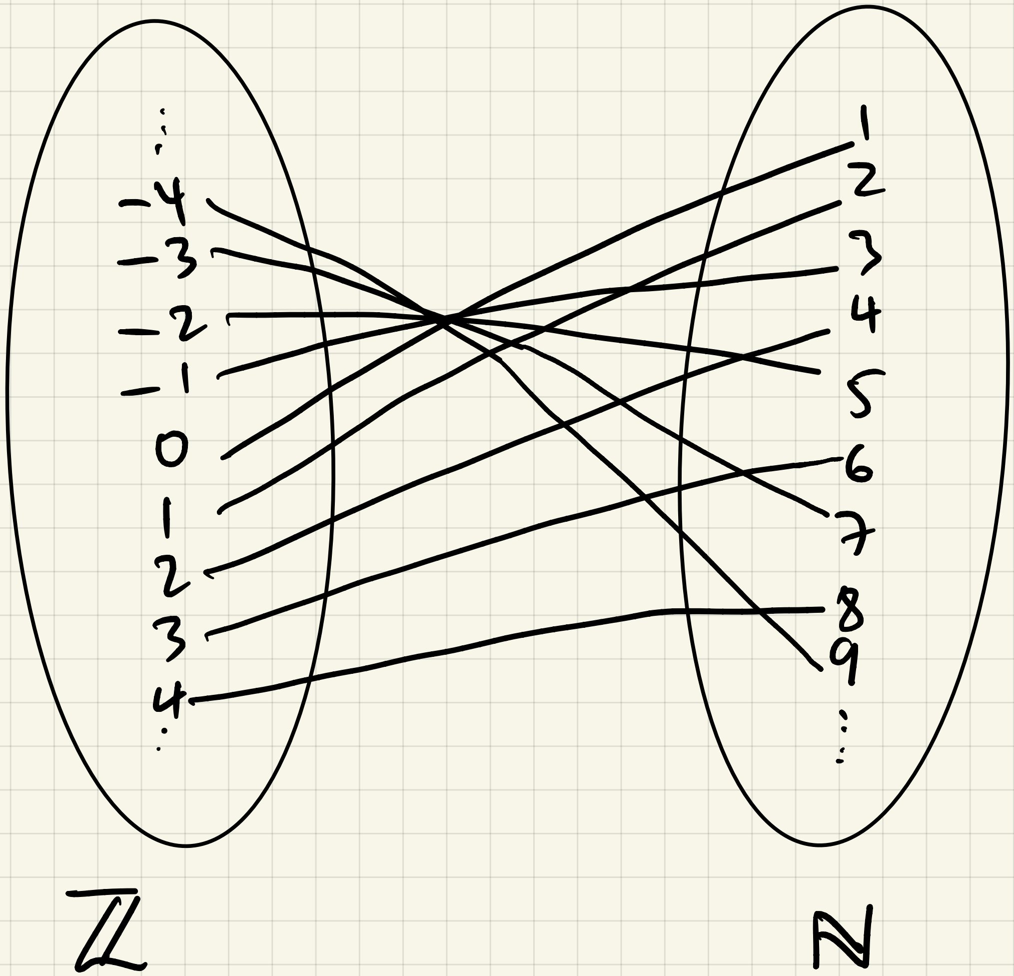
$$\mathbb{Z} = \underbrace{\{0\} \cup \mathbb{N}}_{\text{countable}} \cup \underbrace{\{-1, -2, -3, \dots\}}$$

$f : \{-1, -2, -3, \dots\} \to \mathbb{N}$

$f(x) = -x$

is a bijection

countable



Another way:

Find the bijection

$$f(x) = \begin{cases} 2x & x > 0 \\ -2x+1 & x \leq 0 \end{cases}$$

(Bijection)

Is $\mathbb{Q}$ countable

If A and B are countable, then $A \times B$ is countable

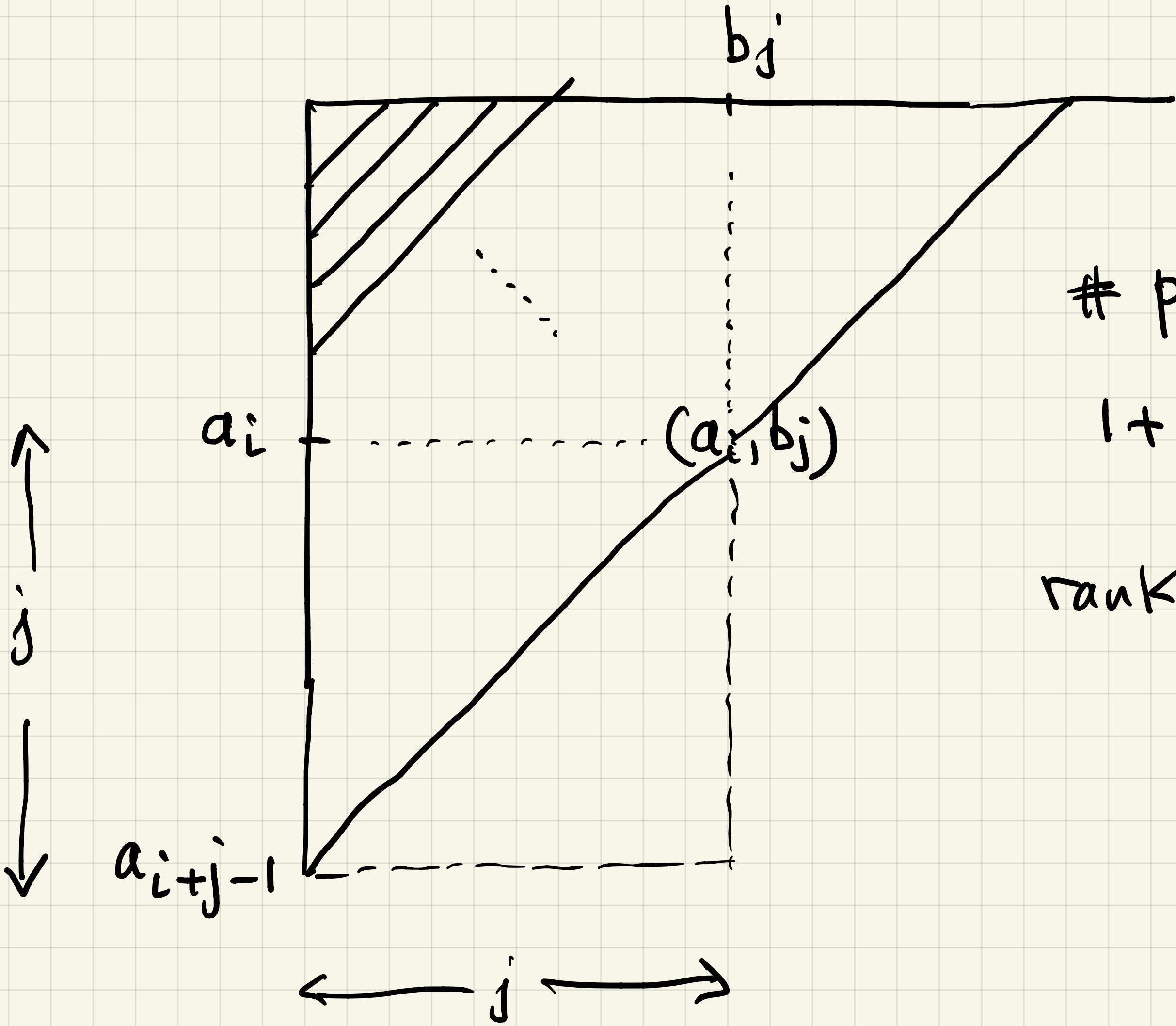
$A = \{ a_1, a_2, a_3, \dots \}$ $\quad$ $B = \{ b_1, b_2, b_3, \dots \}$

	b_1	b_2	b_3	b_4	$\dots$
a_1	(a_1, b_1)	(a_1, b_2)	$\dots$		
a_2	(a_2, b_1)	$\ddots$			
a_3	$\vdots$				
a_4					
$\vdots$					

we can "think" of $\mathbb{Q}$

as a "subset" of $\mathbb{Z} \times \mathbb{N}$

$\frac{a}{b}$ "as" (a, b)



\# pairs in the triangle

$1+2+3+\cdots+(i+j-1)$

rank of $(a_i, b_j) \leqslant 1+2+\cdots+(i+j-1)$

$$= \frac{(i+j-1)(i+j)}{2}$$

$$(i+j-1) - i + 1 = j$$

$\mathbb{R}$ is uncountable (Yay!)

There is NO bijection f

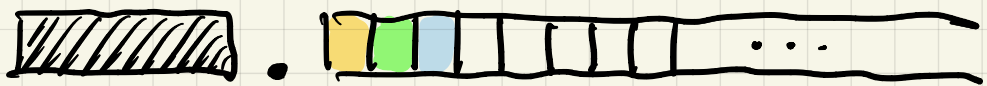
$$f : \mathbb{N} \to \mathbb{R}$$

Proof by contradiction: Cantor's diagonal proof

Let $f : \mathbb{N} \to \mathbb{R}$. We will construct $x \in \mathbb{R}$ such that no $i \in \mathbb{N}$ satisfies $f(i) = x$.

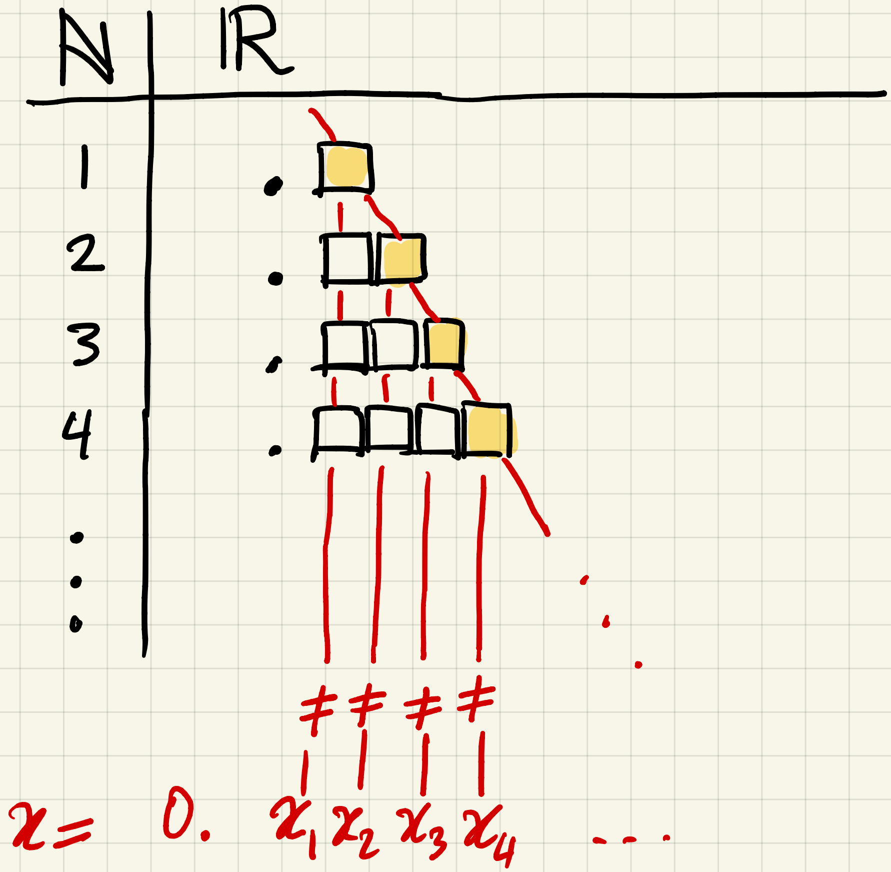
So if f is a bijection, we have a contradiction (it's not onto).

First for every $r \in \mathbb{R}$, we will refer to the i^{th} digit of $\mathbb{R}$ as the i^{th} digit following the decimal point in r's representation.

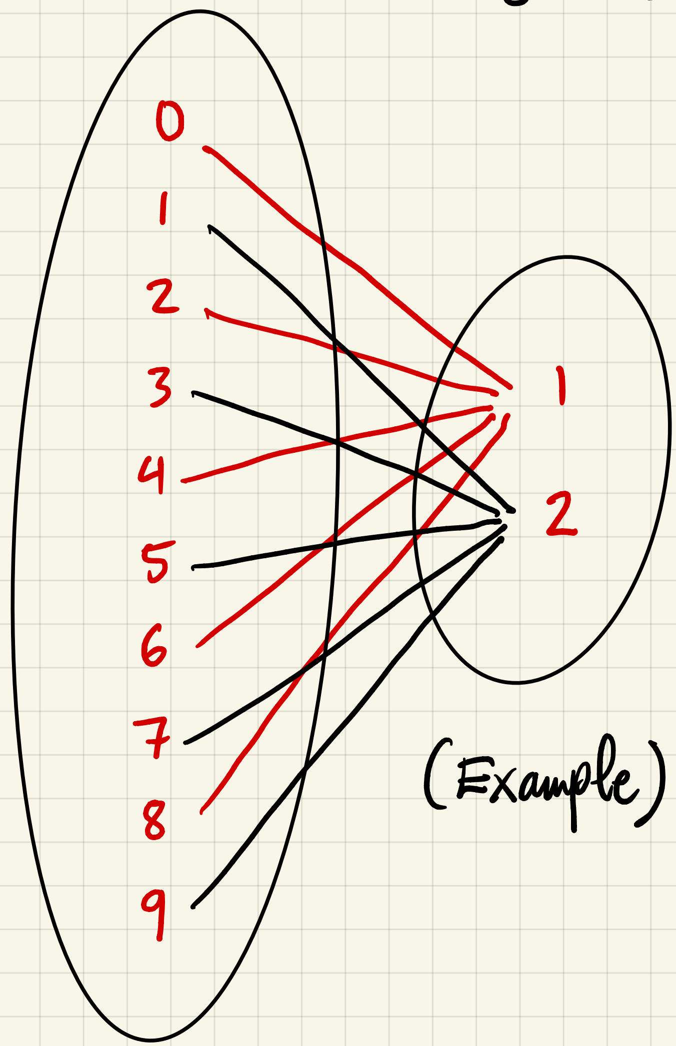
r 

We will make $x = 0.x_1 x_2 x_3 \dots$

where digit $x_i \neq \underbrace{i^{th} \text{ digit of } \overbrace{f(i)}^{\in \mathbb{R}}}_{\text{well defined concept}}$



how to change digits ?



(Example)

There is no $i \in \mathbb{N}$ such that $f(i) = x$ because x is different from $f(i)$ in the ith digit.