Summary:

To show that an infinite set S is Countabe

Formal way: Show there exist $f: S \to \mathbb{N}$ that is a bijection

Informal way: order the elements of S such that each will have a finite rank

Fact: S is countable, any subset of S is countable

A and B are countable, then $A \cup B$, $A \cap B$, $A \times B$ are countable.

$\mathbb{R}$ is uncountable (Yay!)

There is NO bijection f

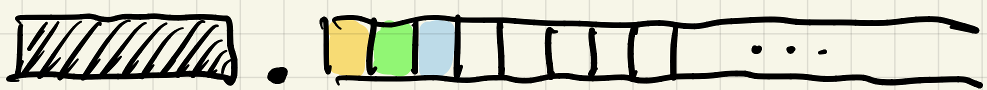
$$f : \mathbb{N} \to \mathbb{R}$$

Proof by Contradiction: Cantor's diagonal proof

Let $f : \mathbb{N} \to \mathbb{R}$. We will construct $x \in \mathbb{R}$ such that no $i \in \mathbb{N}$ satisfies $f(i) = x$.

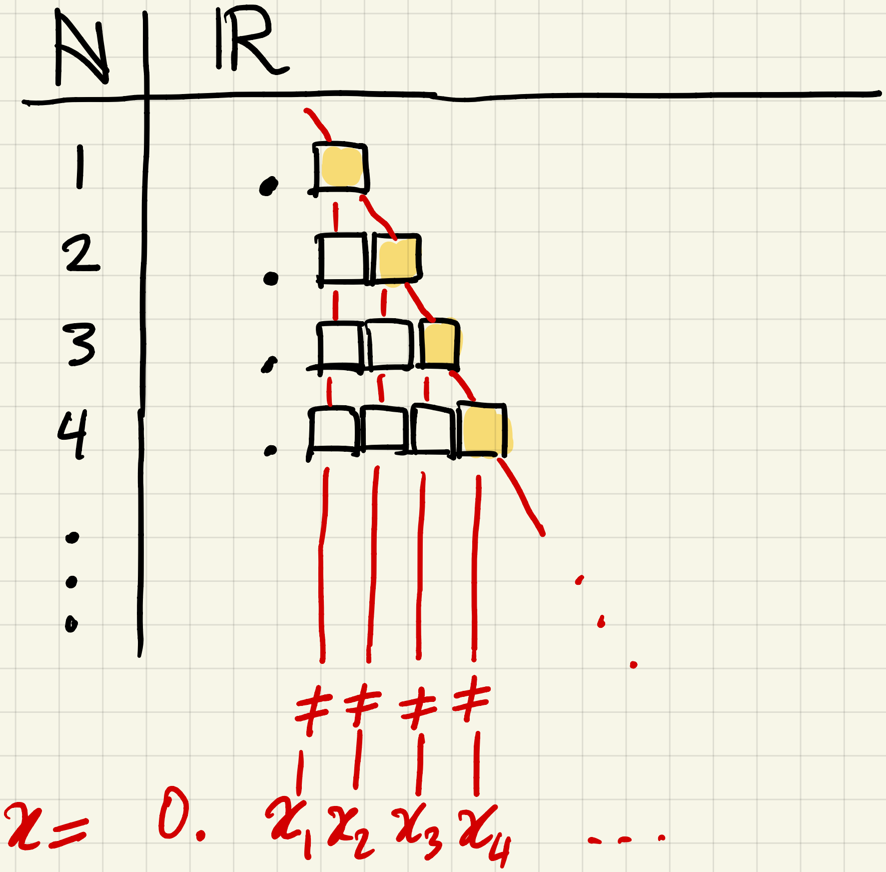
So if f is a bijection, we have a contradiction (it's not onto).

First for every $r \in \mathbb{R}$, we will refer to the i^{th} digit of r as the i^{th} digit following the decimal point in r's representation.

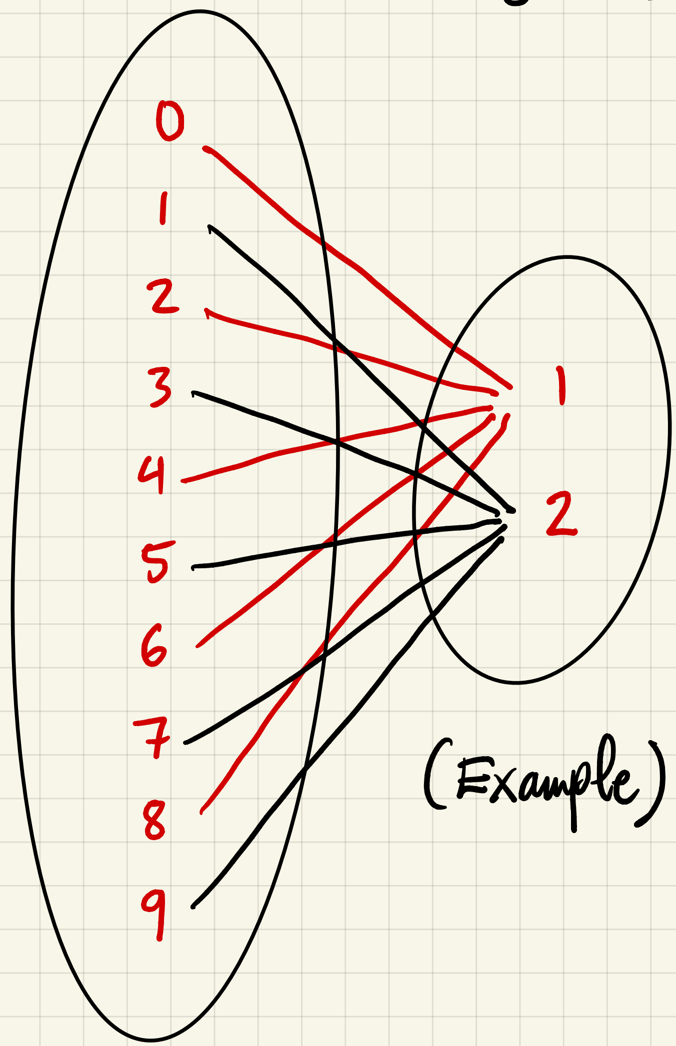
r 

We will make $x = 0.x_1 x_2 x_3 \dots$

where digit $x_i \neq \underbrace{i^{th} \text{ digit of } \overbrace{f(i)}^{\in \mathbb{R}}}_{\text{well defined concept}}$



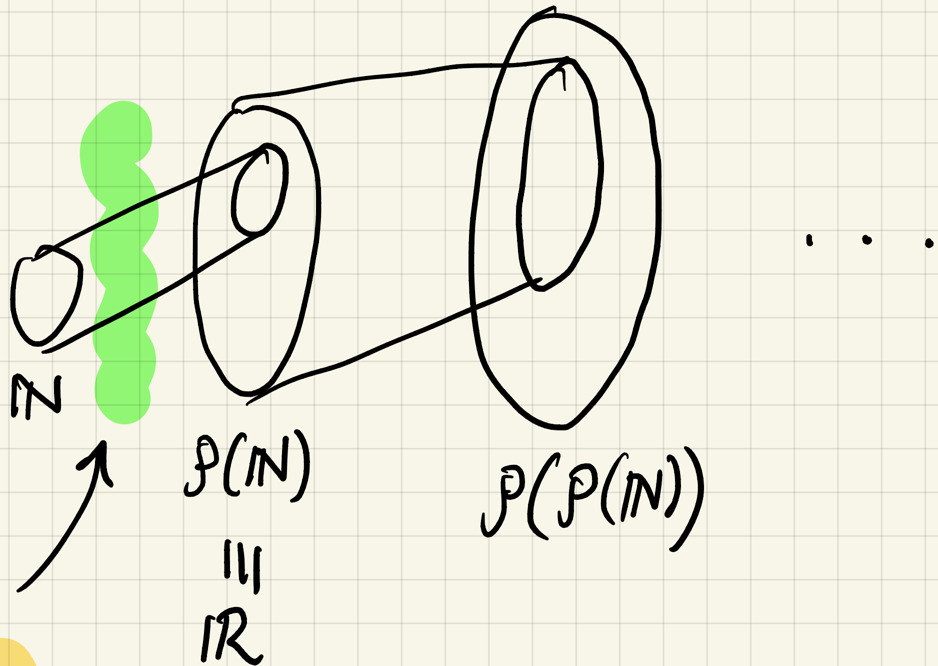
how to change digits ?



(Example)

There is no $i \in \mathbb{N}$ such that $f(i) = x$ because x is different from $f(i)$ in the ith digit.

Given any set S, using a similar diagonal proof we can show there is no bijection from S to $\mathcal{P}(S)$



is there any set in between?

"the continuum hypothesis" is independent of the axioms of set theory (can't prove it or disprove it!)

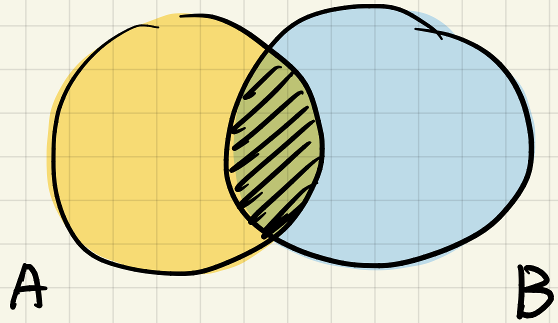
"Example" diagonalization

$\mathbb{N}$	$\mathbb{R}$
1	0.5000000 ...
2	0.1415 ...
3	0.7130000 ...
4	0.8601 00 ...
5	0.3333333 ...

$x = 0.21212 \ldots$

Inclusion - Exclusion

Basic setting:



$$|A \cup B| = |A| + |B| - |A \cap B|$$

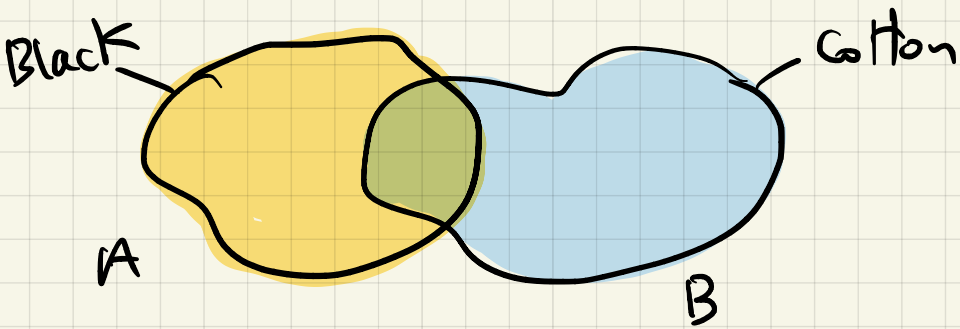
(why ?)

Example: 50 socks

35 black

30 cotton

How many are black and cotton ?

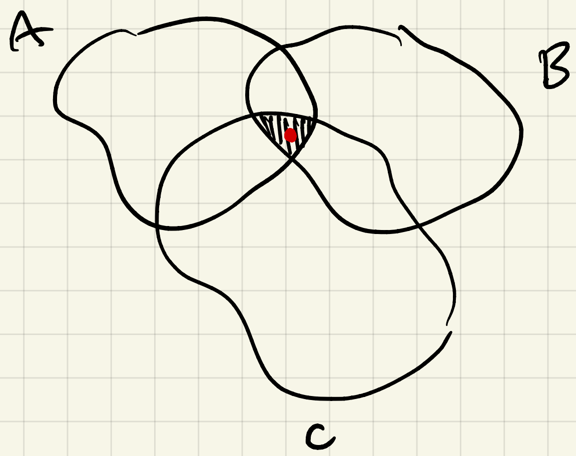


$$|A \cup B| = |A| + |B| - \overbrace{|A \cap B|}^{?}$$

$$50 = 35 + 30 - |A \cap B|$$

$$|A \cap B| = 15$$

What about multiple sets



$$|A \cup B \cup C| = |A| + |B| + |C|$$

$$- |A \cap B| - |A \cap C| - |B \cap C|$$

$$+ |A \cap B \cap C|$$

Four sets:

	# terms
$\lvert A \cup B \cup C \cup D\rvert = \lvert A\rvert + \lvert B\rvert + \lvert C\rvert + \lvert D\rvert$ (include)	$\binom{4}{1}$
$- \lvert A\cap B\rvert - \lvert A \cap C\rvert - \lvert A \cap D\rvert - \lvert B \cap C\rvert - \lvert B \cap D\rvert - \lvert C \cap D\rvert$ (exclude)	$\binom{4}{2}$
$+ \lvert A \cap B \cap C\rvert + \lvert A \cap B \cap D\rvert + \lvert A \cap C \cap D\rvert + \lvert B \cap C \cap D\rvert$ (include)	$\binom{4}{3}$
$- \lvert A \cap B \cap C \cap D\rvert$ (exclude)	$\binom{4}{4}$

Why it works ?

Consider an element that is in n sets.

- It belongs to $\binom{n}{1}$ sets
- It belongs to $\binom{n}{2}$ pairs of sets.
- It belongs to $\binom{n}{3}$ triplets of sets

$\vdots$

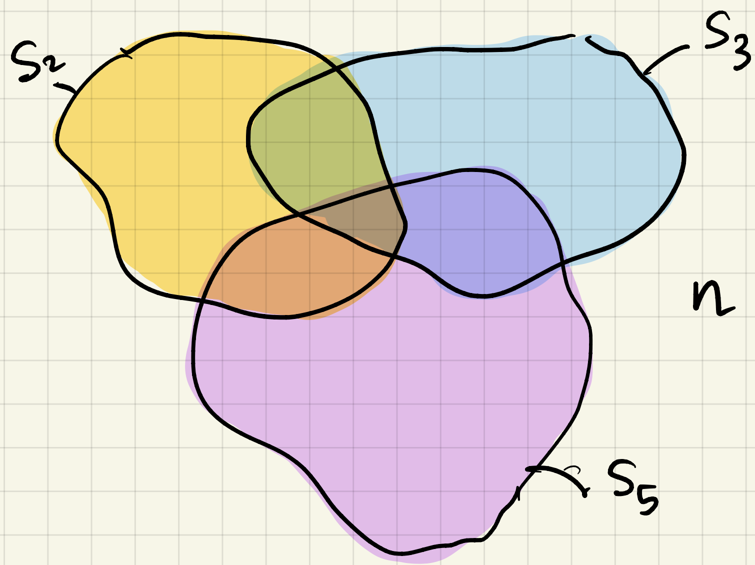
- It belongs to $\binom{n}{n}$ n-tuple of sets

$$\#\text{times element is added} = \binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \binom{n}{4} + \dots + (-1)^{n-1}\binom{n}{n}$$

$$= \binom{n}{0} - \left[\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + (-1)^{n}\binom{n}{n}\right]$$

$$= 1 - 0^n = \begin{cases} n=0: & 1-0^0 = 1-1=0 \\ n>0: & 1-0^n = 1-0 = 1 \end{cases}$$

Example: How many positive integers ≤ 1000 are divisible by 2 or 3 or 5?



if p, q are prime:

n divisible by both $\Longleftrightarrow$ n divisible by pq

$$|S_2 \cup S_3 \cup S_5| = |S_2| + |S_3| + |S_5| - |S_2 \cap S_3| - |S_2 \cap S_5| - |S_3 \cap S_5| + |S_2 \cap S_3 \cap S_5|$$

$$= \left\lfloor \frac{1000}{2} \right\rfloor + \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{5} \right\rfloor - \left\lfloor \frac{1000}{6} \right\rfloor - \left\lfloor \frac{1000}{10} \right\rfloor - \left\lfloor \frac{1000}{15} \right\rfloor + \left\lfloor \frac{1000}{30} \right\rfloor = ?$$