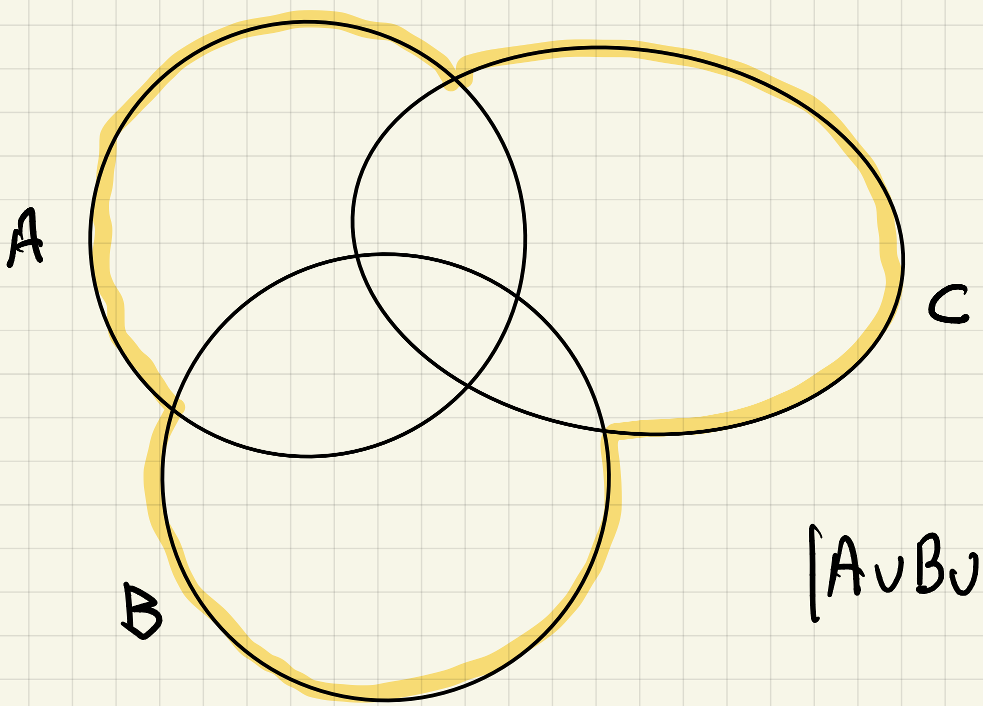
Recall from Inclusion-Exclusion formula



$$|A \cup B \cup C| = |A| + |B| + |C|$$
$$- |A \cap B| - |A \cap C| - |B \cap C|$$
$$+ |A \cap B \cap C|$$

How many passwords of length n are there

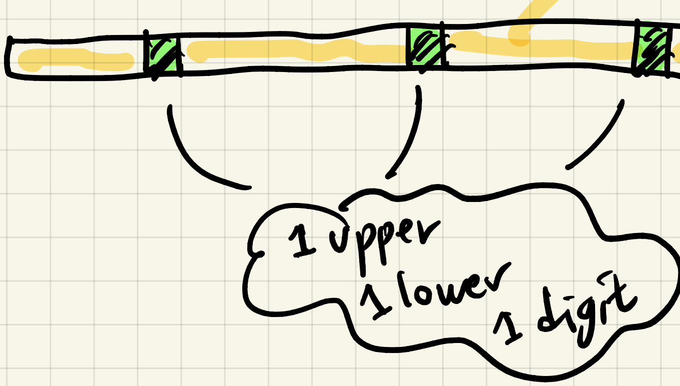
$$\Sigma = \{A, \ldots, Z, a, \ldots, z, 0, \ldots, 9\}$$

Answer: 62^n

<u>Good</u> password has at least one upper case <u>and</u> one lower case <u>and</u> one digit

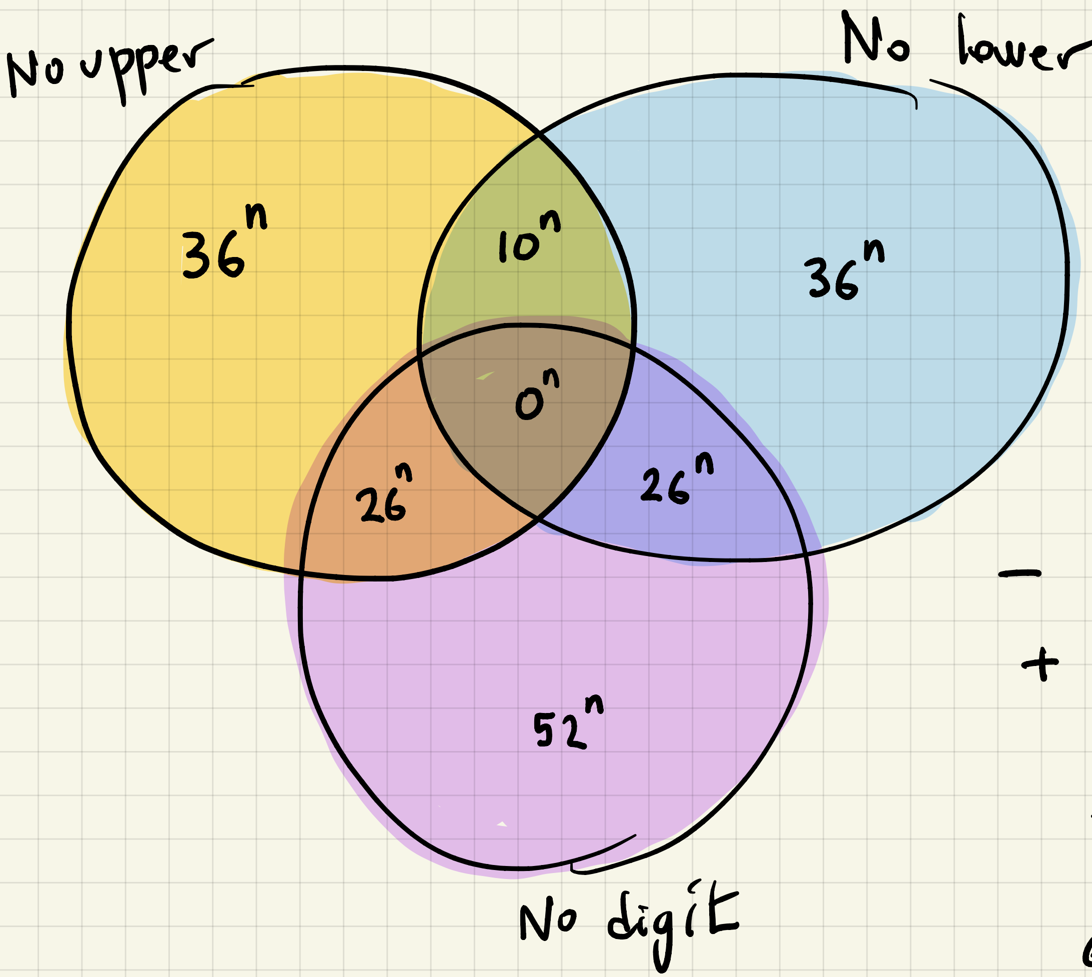
62^{n-3}

product rule overcounts



$$\binom{n}{3} \times 3! \times 26 \times 26 \times 10$$

Bad password



bad words:

$$36^n + 36^n + 52^n - 26^n - 26^n - 10^n + 0^n$$

good words:

$$62^n - (\ \sim\ )$$

Lazy Professor

How many permutations of $(1, 2, 3, \ldots, n)$ are there if every number i does not occur in position i

"Derangements"

This is an "AND" logic.

Initial attempt: (Fails)

ways

1. Choose a position for 1 ---- $(n-1)$
2. Choose a position for 2 --- ?

$(n-2)$?

$(n-1)$?

Can't tell!

Depends on choice for 1

Bad permutation:

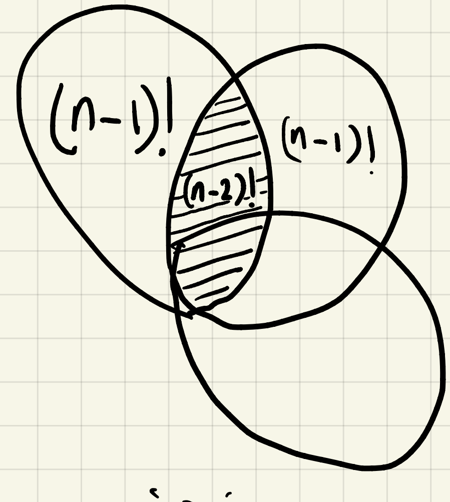
1 is in the first position, **OR**

2 is in the second position, **OR**

3 is " " third position, **OR**

⋮

n is in the n^{th} position.



$$n \cdot (n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! \; \cdots\cdots \; \binom{n}{n}(n-n)!$$

$$\#\text{bad}: \quad \frac{n!}{1!} - \frac{n!}{2!} + \frac{n!}{3!} \; \cdots\cdots \; \frac{n!}{n!}$$

$$\#\text{good}: \quad \frac{n!}{0!} - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots \cdots + (-1)^n \frac{n!}{n!}$$

$$= n! \left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right] \quad \left(e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} \right)$$

$$n \text{ large} \Rightarrow \; \approx n!\, e^{-1} = \frac{n!}{e}$$

derangement:

$$\#\text{derangements} = !n = \left\lfloor \frac{n! + 1}{e} \right\rfloor \qquad n \geq 1$$

$!0 = 1$

$!1 = 0$

$!2 = 1$

$!3 = 2$

$!4 = 9$

$!5 = 44$

$\vdots$

Pigenhole principle

- Choose 51 numbers from $\{1, 2, \dots, 100\}$. Prove that two of them are consecutive.

- Place 10 points inside a 3×3 square. Prove that two of them will be within a distance of $\sqrt{2}$

- Place numbers $1, 2, \dots, 10$ in 3 bins. Prove that sum in one bin is at least 19.

Freedom in doing something, Guaranteed Consequence

Pigeonhole

Pigeon hole: Given $n+1$ objects and n boxes, if we place all objects in boxes, at least one box will contain at least 2 objects.

[Basic form]

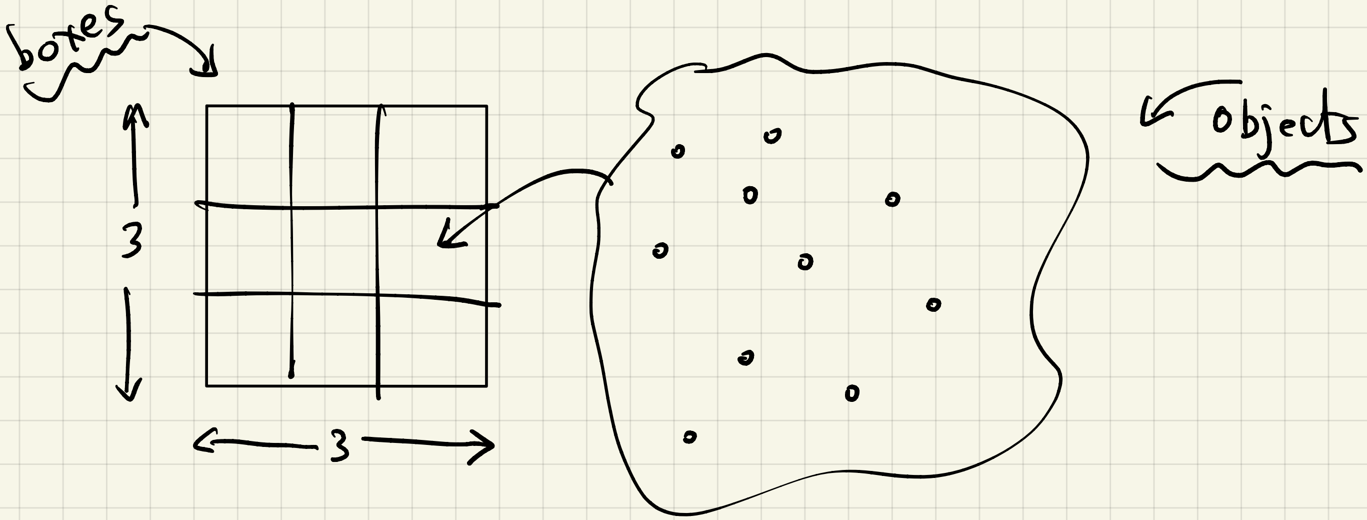
proof: (By contradiction)

Each boxes has at most one objects

Let x_i be # objects in box i, then $x_i \leq 1$

$$\text{Total number of objects} = \sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$
$$\leq 1 + 1 + \cdots + 1 = n$$

a contradiction, since we have $n+1$ objects.



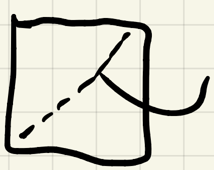
Prove two points will be within a distance $\sqrt{2}$

10 objects
9 boxes

$\Rightarrow$ Pigeonhole : 2 objects will be in the same box

$\Downarrow$

2 points are in the same 1×1 square



longest distance inside square

By Pythagoras, it's $\sqrt{2}$

Choose 51 numbers in $\{1, 2, \dots, 100\}$

Prove 2 of them will be consecutive

Avoid "mehanistic" arguments

1 2 3 4 5 6 7 .... 99 100

can't do it: only 50

Pigenhole argument:

$\boxed{1,2}$ $\boxed{3,4}$ $\boxed{5,6}$ ---- $\boxed{99,100}$

To choose a number, place a token in its box

51 tokens, 50 boxes, done! One box must contain at least 2 tokens $\Rightarrow$ we have 2 consecutive numbers.

A bit more general

Pigeonhole: If we have n boxes, and m objects and we place all objects in boxes, then at least one box will have at least

$$\left\lfloor \frac{m-1}{n} \right\rfloor + 1 \text{ objects.}$$

Proof: similar to previous proof (by contradiction)

If $n, m \in \mathbb{N}$, $\left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lceil \frac{m}{n} \right\rceil$

" some box must contain at least the average "

Place numbers $1, 2, \ldots, 10$ in 3 bins.

- Prove at least one bin has 4 numbers.

$m = 10$, $n = 3$

Pigeonhole: $\left\lceil \frac{m}{n} \right\rceil = \left\lceil \frac{10}{3} \right\rceil = 4$

- Prove at least one bin will have a sum of at least 19.

$m = 1+2+3+\cdots+10 = 55$

$n = 3$

Pigeonhole: $\left\lceil \frac{m}{n} \right\rceil = \left\lceil \frac{55}{3} \right\rceil = 19$