

Proof by Induction

- Proof technique
- Prove something of this form

$$\forall n \in \mathbb{N}, P(n)$$

- Typically, $\mathbb{N} = \{0, 1, 2, 3, \dots\}$

Examples:

- Prove that $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$

$$\forall n \in \mathbb{N}, P(n) \quad \text{where } P(n): \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

- Prove that every positive integer n can be written as

$$n = m 2^k \quad \text{where } m \text{ is odd, } k \in \mathbb{N}$$

- Prove that $\underbrace{1+3+5+\dots+(2n-1)}_{\text{first } n \text{ odds}} = n^2$ for all $n \geq 1$

How to prove by induction

To prove $P(n)$ is true for all $n \geq n_0$ (typically $n_0 = 0$)

- Base case: Prove $P(n_0)$ is true (verification)

- Inductive step: Prove $\forall k \geq n_0, P(k) \Rightarrow P(k+1)$

(Assume $P(k)$ is true, use that to prove $P(k+1)$ is true)

This establishes: $\forall n \geq n_0, P(n)$ is true

Example 1: Prove $\sum_{i=0}^n 2^i = 2^{n+1} - 1$ for all $n \geq 0$

Base case: $n_0 = 0$: $P(0)$: $\sum_{i=0}^0 2^i = 2^{0+1} - 1$

$$2^0 = 2^1 - 1$$

$$1 = 1 \quad \checkmark$$

Inductive step: $\forall k \geq 0, P(k) \Rightarrow P(k+1)$

$P(k)$: $\sum_{i=0}^k 2^i = 2^{k+1} - 1$ (Inductive hypothesis)

$P(k+1)$: $\sum_{i=0}^{k+1} 2^i = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$

$$\begin{aligned} \sum_{i=0}^{k+1} 2^i &= \sum_{i=0}^k 2^i + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} \\ &= 2^{k+1} + 2^{k+1} - 1 = 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+2} - 1 \quad \text{Done} \end{aligned}$$

Example 2 $\prod_{i=1}^n \left(1 + \frac{1}{i}\right) = n+1$ for all $n \geq 1$.

Base case: $n_0 = 1$. $\prod_{i=1}^1 \left(1 + \frac{1}{i}\right) = 1+1$
 $1 + \frac{1}{1} = 1+1 \quad \checkmark$

Inductive step: $\forall k \geq 1, P(k) \implies P(k+1)$

$P(k): \prod_{i=1}^k \left(1 + \frac{1}{i}\right) = k+1$ (Inductive hypo)

$P(k+1): \prod_{i=1}^{k+1} \left(1 + \frac{1}{i}\right) = (k+1) + 1 = k+2$

$$\prod_{i=1}^{k+1} \left(1 + \frac{1}{i}\right) = \prod_{i=1}^k \left(1 + \frac{1}{i}\right) \times \left(1 + \frac{1}{k+1}\right) = (k+1) \left(1 + \frac{1}{k+1}\right) = (k+1) + 1 = k+2.$$

Example 3

$$T_0 = 0$$

$$T_{n+1} = T_n \overline{T_n} \quad (\overline{T_n} = T_n \text{ with bits negated})$$

$$T_1 = T_0 \overline{T_0} = 01$$

$$T_2 = T_1 \overline{T_1} = 0110$$

$$T_3 = T_2 \overline{T_2} = 01101001$$

Prove T_n has 2^n bits for all $n \in \mathbb{N} = \{0, 1, 2, \dots\}$

Base case: $n_0 = 0$. $P(0)$: T_0 has 2^0 bits ✓

Inductive step: $\forall k \geq 0, P(k) \Rightarrow P(k+1)$

$P(k)$: T_k has 2^k bits (Ind-hyp)

$P(k+1)$: T_{k+1} has 2^{k+1} bits

$$T_{k+1} = T_k \overline{T_k} \text{ which has } 2^k + 2^k \text{ bits} = 2 \cdot 2^k \text{ bits} = 2^{k+1} \text{ bits.}$$

Done.

Example 4: Prove that T_n starts with 01 for all $n \geq 1$

Base case: $n_0 = 1$. $T_1 = 01$ so it starts with 01 ✓

Inductive step: $\forall k \geq 1, P(k) \Rightarrow P(k+1)$

$P(k)$: T_k starts with 01 (Inductive hyp)

$P(k+1)$: T_{k+1} starts with 01

$T_{k+1} = T_k \overline{T_k}$ and therefore T_{k+1} starts with 01 since
 T_k starts with 01. Done.

Example 5. Prove T_n ends in 10 if n even
and 01 if n odd for $n \geq 1$

Base case: $n_0 = 1$. $T_1 = 01$, it ends in 01 and 1 is odd ✓

Inductive step: $\forall k \geq 1, P(k) \Rightarrow P(k+1)$

$$T_{k+1} = T_k \overline{T_k}$$

$k+1$ even $\Rightarrow k$ is odd $\Rightarrow T_k$ ends in 01 (Inductive hyp.)

$\Rightarrow \overline{T_k}$ ends in 10 $\Rightarrow T_{k+1}$ ends in 10.

$k+1$ is odd $\Rightarrow \dots$ (symmetric argument)

Example 6 Prove $n^3 - n$ is a multiple of 3 for all $n \geq 0$

Base case: $n_0 = 0$. $P(0) : 0^3 - 0 = 0$ is multiple of 3 ✓

Inductive step: $\forall k \geq 0, P(k) \Rightarrow P(k+1)$

$P(k) : k^3 - k = 3m$ (Inductive hypothesis)

$P(k+1) : (k+1)^3 - (k+1) = 3m'$ $m, m' \in \mathbb{Z}$

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1$$

$$= (k^3 - k) + 3(k^2 + k)$$

Ind. hyp.

$$= 3m + 3(k^2 + k) = 3 \underbrace{(m + k^2 + k)}_{m'}. \text{ Done}$$

What's wrong with this proof?

$$\text{Prove } \sum_{i=1}^n i = \frac{n^2 + n + \sqrt{\pi}}{2} \quad \forall n \in \mathbb{N}$$

$$P(k): \sum_{i=1}^k i = \frac{k^2 + k + \sqrt{\pi}}{2}$$

$$P(k+1): \sum_{i=1}^{k+1} i = \frac{(k+1)^2 + (k+1) + \sqrt{\pi}}{2}$$

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^k i + (k+1) = \frac{k^2 + k + \sqrt{\pi}}{2} + \frac{2(k+1)}{2} = \dots = \frac{(k+1)^2 + (k+1) + \sqrt{\pi}}{2}$$

Done.