For every $n \geqslant 12$, $n = 3x + 7y$ where $x, y \in \mathbb{N}$

Base case: $n_0 = 12$. $P(12)$: $12 = 3 \times 4 + 7 \times 0$ ✓

Inductive step: $\forall\, k \geqslant n_0$, $P(k) \Rightarrow P(k+1)$

$P(k)$: $k = 3x + 7y$, $x, y \in \mathbb{N}$

$P(k+1)$: $k+1 = 3x' + 7y'$, $x', y' \in \mathbb{N}$.

$$k+1 = \underbrace{[3x + 7y]}_{k} + \underbrace{7 - 2 \cdot 3}_{1} = 3\underbrace{(x-2)}_{\substack{x' \\ \in \mathbb{N}?}} + 7\underbrace{(y+1)}_{\substack{y' \\ \in \mathbb{N}}}$$

✗

Not necessarily

Strong Induction

Base case: Prove $P(k)$ is true for $k \leq n_0$

Inductive Step: $\forall\, k \geq n_0,\ \underbrace{\bigwedge_{i \leq k} P(i)}_{\text{Ind. hypo}} \Longrightarrow P(k+1)$

Assume the property is true up to k, then prove the property is true at $k+1$.

Notation: $\bigwedge_{i \leq k} P(i) = P(k) \wedge P(k-1) \wedge P(k-2) \wedge \dots$

Typically only a few value in $\{k, k-1, k-2, \dots\}$ are used for the inductive hypo.

Example 1. For $n \geq 12$, $n = 3x + 7y$ where $x, y \in \mathbb{N}$

Base case: $P(12)$: $12 = 3 \times 4 + 7 \times 0$ ✓

$P(13)$: $13 = 3 \times 2 + 7 \times 1$

$P(14)$: $14 = 3 \times 0 + 7 \times 2$

Inductive step: $P(12), P(13), P(14), \ldots, P(k)$ are true

$P(k+1)$: $k+1 = 3x' + 7y'$

$$k+1 = (k+1) + 3 - 3 = \underbrace{(k-2)}_{} + 3$$

Ind. hypo. $\to$ $3x + 7y + 3 = 3\underbrace{(x+1)}_{\in \mathbb{N}} + 7\underbrace{y}_{\in \mathbb{N}}$

The proof works when $k - 2 \geq 12 \Rightarrow k \geq 14$

Example 2. Every $n \geq 1$, $n = m \cdot 2^i$ where m is odd.

Base case: $n = 1$. $1 = 1 \times 2^0$ ✓

Inductive step: $P(1), P(2), \ldots, P(k)$ are true

$P(k+1)$: $k+1 = m \cdot 2^i$ (m odd)

- $k+1$ is odd: $k+1 = (k+1) \cdot 2^0$
- $k+1$ is even: $k+1 = 2j$ where $j \leq k$

$P(j)$ is true by inductive hypo.

$$j = m' \cdot 2^{i'}$$

$$k+1 = 2j = m' \cdot 2^{i'} \cdot 2 = m' \cdot 2^{(i'+1)} = m \cdot 2^i$$

$$j = \frac{k+1}{2} \leq k \Rightarrow k+1 \leq 2k \Rightarrow k \geq 1.$$

Example 3

Consider: $a_1, a_2, a_3, \dots$

$$a_1 = 3$$
$$a_2 = 5$$
$$a_n = 3a_{n-1} - 2a_{n-2}, \quad n \geq 3$$

Let's find few of these terms:

$$a_3 = 3 \times a_2 - 2a_1 = 9$$
$$a_4 = 3 \times a_3 - 2a_2 = 17$$
$$a_5 = 3 \times a_4 - 2a_3 = 33$$
$$\vdots$$

Guess what a_n is? $\quad a_n = 2^n + 1$

Prove by induction.

Prove $a_n = 2^n + 1$ for all $n \geq 1$

Base case: $a_1 = 2^1 + 1 = 3$ ✓

$a_2 = 2^2 + 1 = 5$ ✓

Inductive step: $P(1), P(2), \ldots, P(k)$ are true

Ind. hypo.

$$
\begin{aligned}
P(k+1) : a_{k+1} &= 3a_k - 2a_{k-1} \\
&= 3(2^k + 1) - 2(2^{k-1} + 1) \\
&= 3 \times 2^k + 3 - 2 \times 2^{k-1} - 2 \\
&= 3 \times 2^k - 2^k + 1 \\
&= 2 \times 2^k + 1 \\
&= 2^{k+1} + 1
\end{aligned}
$$

This proof works when both a_k and a_{k-1} exist.

$k - 1 \geq 1 \Rightarrow k \geq 2$

Example 4   Fibonacci Sequence

$$F_0 = 0$$

$$F_1 = 1$$

$$F_n = F_{n-1} + F_{n-2} \qquad n \geq 2$$

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

Prove $\quad F_n = \frac{1}{\sqrt{5}}\left[\phi^n - (1-\phi)^n\right] \quad$ for $n \geq 0$

where $\phi = \frac{1+\sqrt{5}}{2}$ (Golden Ratio)

Base case: $F_0 = \frac{1}{\sqrt{5}}\left[\phi^0 - (1-\phi)^0\right] = \frac{1}{\sqrt{5}}\left[1 - 1\right] = 0$ ✓

$F_1 = \frac{1}{\sqrt{5}}\left[\phi^1 - (1-\phi)^1\right] = \frac{1}{\sqrt{5}}(2\phi - 1) = 1$ ✓

Inductive step: $P(0), P(1), P(2), \ldots, P(k)$

$$P(k+1) : F_{k+1} = \frac{1}{\sqrt{5}}\left[\phi^{k+1} - (1-\phi)^{k+1}\right]$$

$$F_{k+1} = F_k + F_{k-1}$$

$$= \frac{1}{\sqrt{5}}\left[\phi^{k} - (1-\phi)^{k}\right] + \frac{1}{\sqrt{5}}\left[\phi^{k-1} + (1-\phi)^{k-1}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\phi^{k} + \phi^{k-1}\right] - \frac{1}{\sqrt{5}}\left[(1-\phi)^{k} + (1-\phi)^{k-1}\right]$$

$$= \frac{1}{\sqrt{5}}\phi^{k+1}\left[\underbrace{\frac{1}{\phi} + \frac{1}{\phi^2}}_{1}\right] - \frac{1}{\sqrt{5}}(1-\phi)^{k+1}\left[\underbrace{\frac{1}{1-\phi} + \frac{1}{(1-\phi)^2}}_{1}\right]$$

Both ϕ and $(1-\phi)$ are solutions to $\frac{1}{x} + \frac{1}{x^2} = 1$

$$x + 1 = x^2$$