Today: How recurrences help with counting

- Instead of counting the exact thing
  Find a recurrence for it
- At the price of having to solve recurrence

solve a recurrence

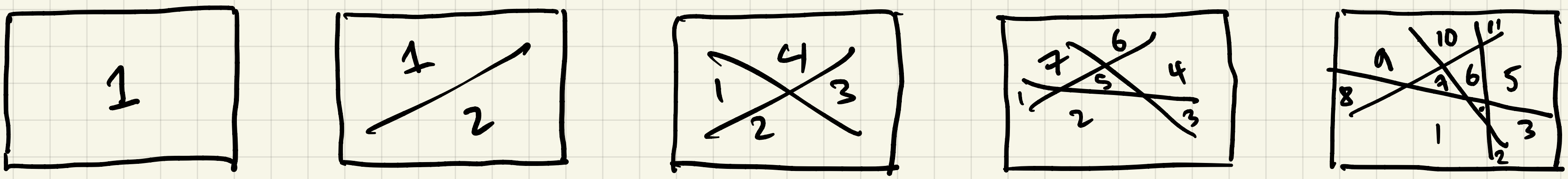
. Asymptotically
(not exact)

. Find $a_0, a_1, a_2, a_3 \ldots$
. Guess a pattern of a_n
. Prove it by induction

. Put recurrence in a form
you know
eg, $a_n = Aa_{n-1} + Ba_{n-2}$
. Solve using characteristic
equation

. Generating functions
(maybe later)

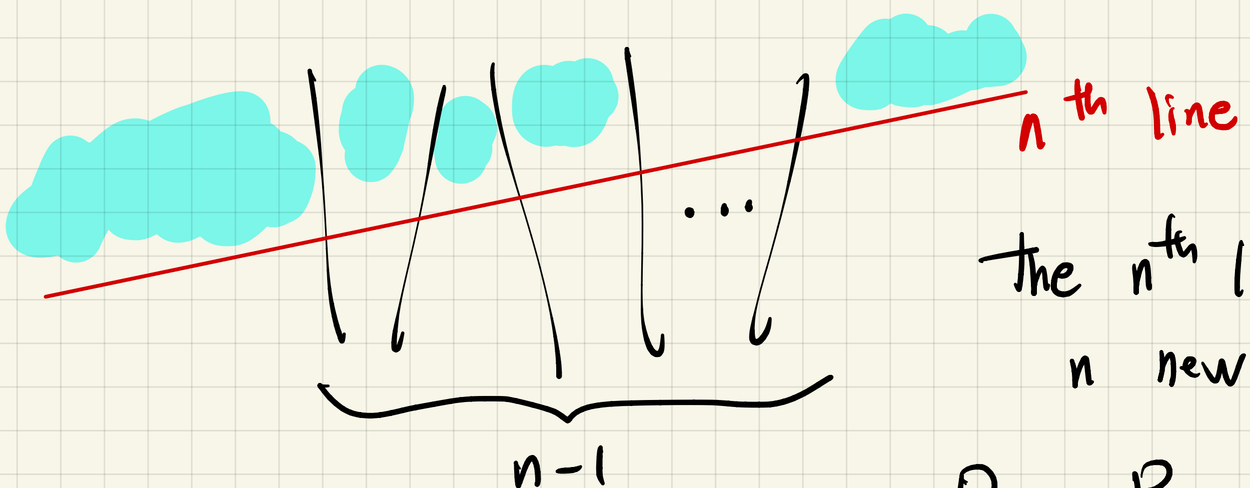
Example 1: How many regions n lines make in the plane if no two are parallel and no three intersect in one point



0 lines   1 line   2 lines   3 lines   4 lines

let R_n = # regions made by n lines (Name It!!)

$R_0 = 1$, $R_1 = 2$, $R_2 = 4$, $R_3 = 7$, $R_4 = 11$



The n^{th} line adds n new regions

$$R_n = R_{n-1} + n$$

n	0	1	2	3	4	5	6	...
	1	2	4	7	11	16	22	...

Example:

$$11 = 4 + 7$$
$$3 + 4$$
$$2 + 2$$
$$1 + 1$$

$$R_n = (1 + 2 + \cdots + n) + 1$$

$$R_n = \frac{n(n+1)}{2} + 1$$

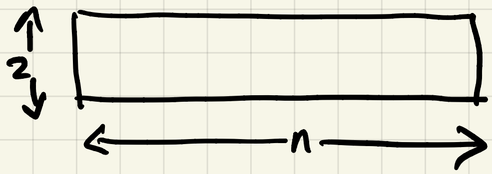
Prove it by induction.

Base case: $R_0 = \dfrac{0(0+1)}{2} + 1 = 1$ ✓

Inductive Step: $\forall\, k \geq 1,\ P(k) \Rightarrow P(k+1)$

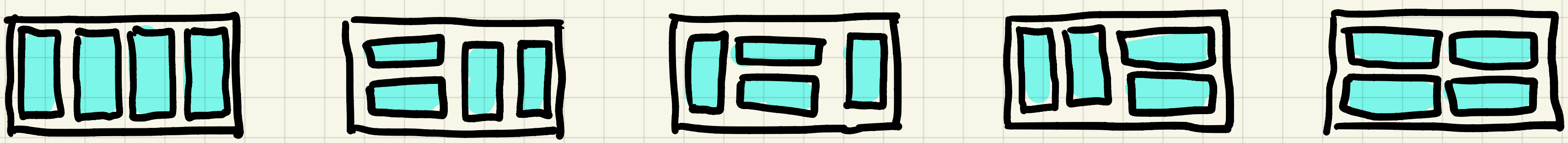
$$R_{k+1} = R_k + (k+1) = \underbrace{\frac{k(k+1)}{2} + 1}_{\text{ind. hypo.}} + k+1 = \cdots = \frac{(k+1)(k+2)}{2} + 1.$$

Example 2. Tiling a $2 \times n$ rectangle by dominos.



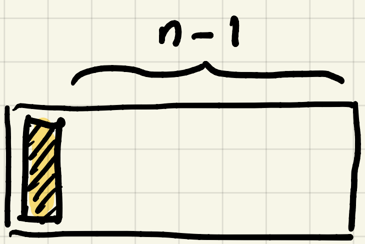
In how many ways can we tile a $2 \times n$ rectangle with dominos

$n = 4$:

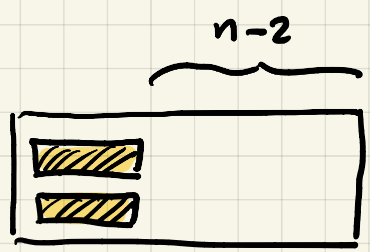


Let a_n be # ways we can tile a rectangle of length n $(a_4 = 5)$

Two cases



a_{n-1} ways of finishing



a_{n-2} ways of finishing

$$a_n = a_{n-1} + a_{n-2}$$

$a_0 = 1$ $\qquad F_0 = 0$

$a_1 = 1$ $\qquad F_1 = 1$

$a_2 = 2$ $\qquad F_2 = 1$

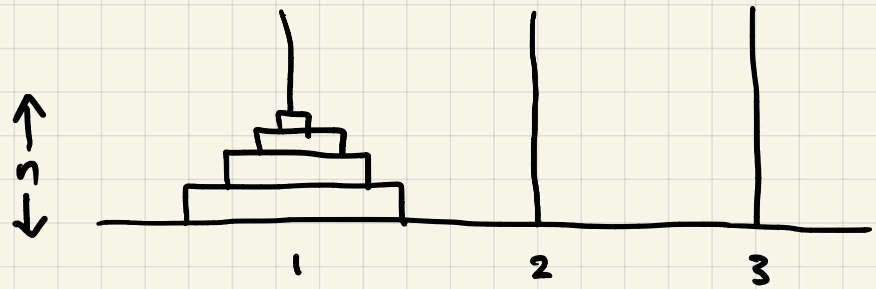
$a_3 = 3$ $\qquad F_3 = 2$

$a_4 = 5$ $\qquad F_4 = 3$

$\vdots$ $\qquad F_5 = 5$

$$a_n = F_{n+1}$$

Example 3: Tower of Hanoi



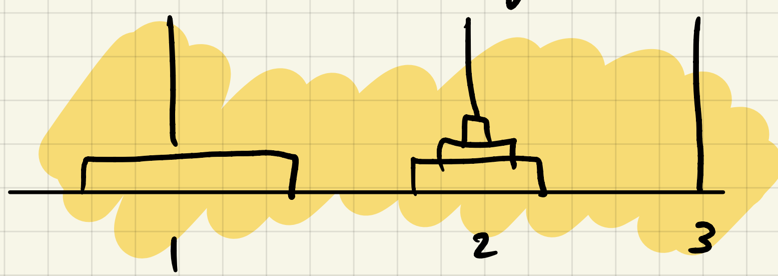
Move n disks from peg 1 to peg 3

- move one disk at a time
- no disk can sit on top of a smaller one

How many moves are needed to transfer a pile of n disks?

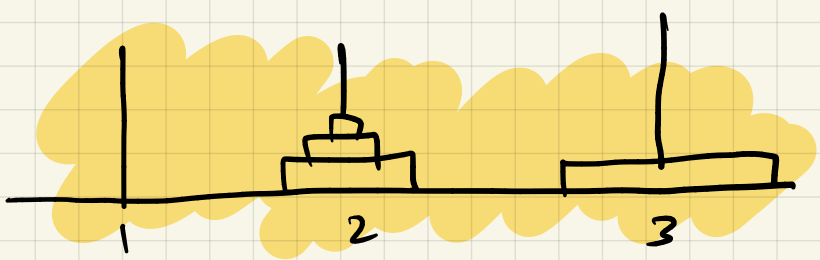
Let a_n = # moves to transfer n disks.

In order to move the largest disk, we must reach this:



I must have moved $n-1$ disks with a_{n-1} moves

Then make 1 move



Then I need another a_{n-1} moves

$$a_n = a_{n-1} + 1 + a_{n-1}$$

$$a_n = 2a_{n-1} + 1$$

- We solved this recurrence by eliminating 1 and putting it in the form $a_n = A a_{n-1} + B a_{n-2}$
- Another way: Establish a pattern

$$a_0 = 0 \quad a_1 = 1 \quad a_2 = 3 \quad a_3 = 7 \quad a_4 = 15 \quad a_5 = 31 \quad \ldots$$

$$a_n = 2^n - 1$$

Generating Functions

The generating function of the sequence

$$a_0, a_1, a_2, a_3, \ldots$$

is $f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \ldots$

$$= \sum_{i=0}^{\infty} a_i x^i$$

Example: Generating function for Fibonacci

$$f(x) = 0 \cdot x^0 + 1 \cdot x^1 + 1 \cdot x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \ldots$$

The n^{th} derivative of $f(x)$ at $x=0$ divided by $n!$ is a_n

$$a_n = \frac{f^{(n)}(0)}{n!} \quad \text{(for any sequence)}$$

$$f(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots$$

$$f(0) = a_0 \Rightarrow \frac{f(0)}{0!} = a_0$$

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots$$

$$f'(0) = a_1 \Rightarrow \frac{f'(0)}{1!} = a_1$$

$$f''(x) = 2a_2 + 6a_3x + \cdots$$

$$f''(0) = 2a_2 \Rightarrow \frac{f''(0)}{2!} = a_2$$

$$f'''(x) = 6a_3 + \cdots$$

$$f'''(0) = 6a_3 \Rightarrow \frac{f'''(0)}{3!} = a_3$$

$$\vdots$$

So $$f(x) = \sum_{i=0}^{\infty} \underbrace{\frac{f^{(i)}(0)}{i!}}_{a_i} x^i$$ (Maclaurin series)