Generating Functions

The generating function of the sequence

$$a_0,\ a_1,\ a_2,\ a_3,\ \ldots$$

is $f(x) = a_0x^0 + a_1x^1 + a_2x^2 + \ldots$

$$= \sum_{i=0}^{\infty} a_i x^i$$

<u>Example:</u> Generating function for Fibonacci

$$f(x) = 0.x^0 + 1.x^1 + 1.x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \ldots$$

The n^{th} derivative of $f(x)$ at $x=0$ divided by $n!$ is a_n

$$a_n = \frac{f^{(n)}(0)}{n!} \quad \text{(for any sequence)}$$

$$f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots$$

$$f(0) = a_0 \Rightarrow \frac{f(0)}{0!} = a_0$$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$f'(0) = a_1 \Rightarrow \frac{f'(0)}{1!} = a_1$$

$$f''(x) = 2a_2 + 6a_3 x + \cdots$$

$$f''(0) = 2a_2 \Rightarrow \frac{f''(0)}{2!} = a_2$$

$$f'''(x) = 6a_3 + \cdots$$

$$f'''(0) = 6a_3 \Rightarrow \frac{f'''(0)}{3!} = a_3$$

$$\vdots$$

So $$f(x) = \sum_{i=0}^{\infty} \underbrace{\frac{f^{(i)}(0)}{i!}}_{a_i} x^i$$ (Maclaurin series)

Example:

$$a_n = 3\times 2^{n-1} - a_{n-1}, \quad a_1 = 0$$

$$a_1 = 0, \quad a_2 = 3\times 2^{2-1} - a_1 = 6, \quad \dots$$

Solution: $a_n = 2^n + 2(-1)^n$. [look for it]

Generating function:

$$f(x) = a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_1 x^1 + (3\times 2 - a_1)\, x^2 + (3\times 2^2 - a_2)\, x^3 + (3\times 2^3 - a_3)\, x^4 + \cdots$$

$$= a_1 x + 6x^2(1 + 2x + 4x^2 + \cdots) - (a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots)$$

$$f(x) = 6x^2 \cdot \frac{1}{1-2x} - x f(x)$$

$$\left\{ 1 + a + a^2 + a^3 + \cdots = \frac{1}{1-a} \quad \text{if } |a| < 1 \right\}$$

$$f(x) = \frac{6x^2}{(1+x)(1-2x)}$$

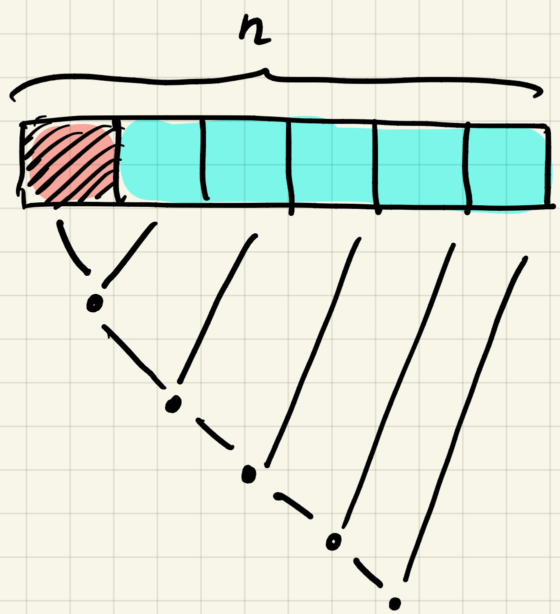
$$f(x) = 2x^2 \left[\frac{1}{1+x} + \frac{2}{1-2x} \right]$$

$$f(x) = 2x^2 \left[1 - x + x^2 - x^3 + \dots \right] + 4x^2 \left[1 + 2x + 4x^2 + \dots \right]$$

Coefficient of x^n is $\underbrace{4 \cdot 2^{n-2} + 2(-1)^n}_{a_n}$

$$a_n = 2^n + 2(-1)^n$$

Sorting an array



$(n-1)$ comparisions.

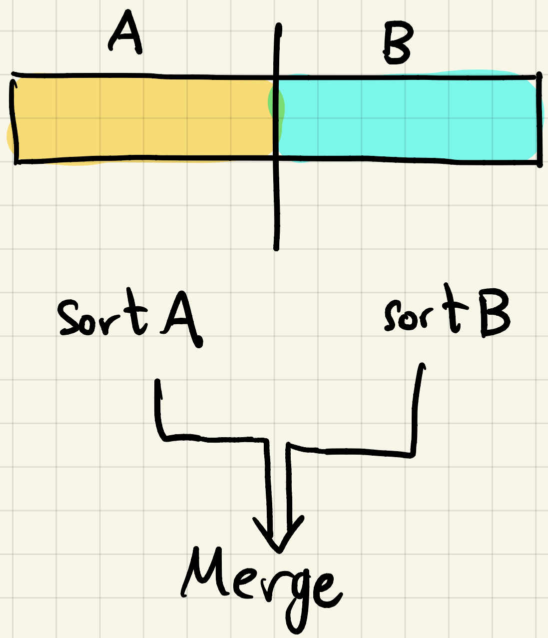
Repeatedly find the smallest element, move it to the beginning of array.

total: $(n-1) + (n-2) + (n-3) + \cdots + 1 = \dfrac{n(n-1)}{2} \approx \dfrac{n^2}{2}$

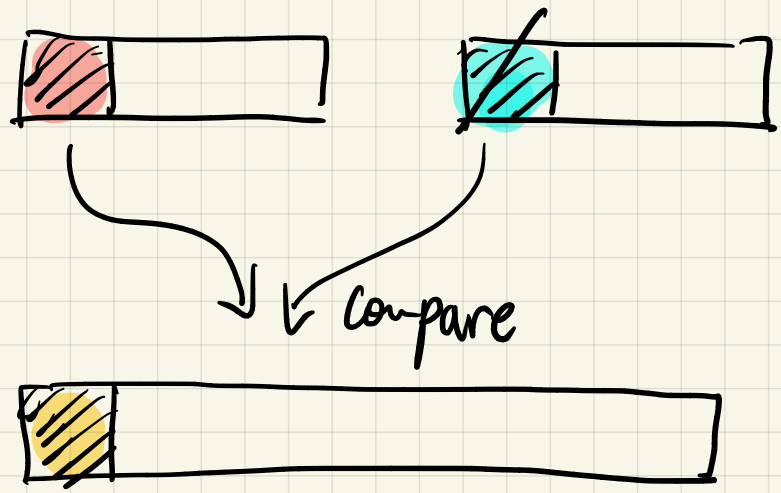
$$T_n = (n-1) + T_{n-1}, \quad T_1 = 0$$

A better idea

Merge Sort : Divide array into two "equal" size arrays



(Merging ...)



$$T_n = (n-1) + T_{\lfloor n/2 \rfloor} + T_{\lceil n/2 \rceil}$$

move smaller of the two

$(n-1)$ comparisons.

[each comparison move an elem]

Approximation: $T_n = n + 2T_{n/2}$

(Assume n is a power of 2) $\quad T_1 = 0$

$T_1 \quad T_2 \quad T_4 \quad T_8 \quad T_{16}$

$a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4$

$a_n = T_{2^n}$

$$a_n = T_{2^n} = 2^n + 2T_{2^n/2} = 2^n + 2T_{2^{n-1}}$$

$$\boxed{a_n = 2^n + 2a_{n-1}} \qquad a_0 = 0 \qquad a_1 = 2$$

We have seen this before

solution: $a_n = n2^n$

$$a_n = T_{2^n} \iff T_n = a_{\log_2 n} = \log_2 n \cdot 2^{\log_2 n} = n \log_2 n .$$

Number Theory

Divisibility: Definition & Notation

1. a divides b
2. a is a divisor of b
3. b is a multiple of a

   $\exists\, m \in \mathbb{Z},\quad b = ma$ (definition)

4. $a \mid b$ (notation)

If a does not divide b $(a \nmid b)$

In general,

[unique representation] $b = a \cdot q + r$ where $0 \leqslant r < a$

$(r = 0 \Rightarrow a \mid b)$

q: quotient

r: remainder, $r \in \{0, 1, 2, \ldots, a-1\}$

Prove uniqueness: (By contradiction)

Suppose $b = aq_1 + r_1 = aq_2 + r_2$ $(r_2 > r_1)$

what can we say about $r_2 - r_1$?

$$0 < r_2 - r_1 < a$$

$$r_2 = b - aq_2$$

$$r_1 = b - aq_1$$

$$r_2 - r_1 = (b - aq_2) - (b - aq_1)$$

$$= a(q_1 - q_2)$$

Now, $0 < a(q_1 - q_2) < a$

$0 < q_1 - q_2 < 1$, a contradiction.

because there is no integer strictly between 0 and 1.

Given two integers a and b, the <u>greatest common divisor</u> of a and b

$$\gcd(a, b)$$

is a divisor of a and a divisor of b and it's the largest such integer.

Well defined Concept:

- 1 is a common divisor, so there is one
- Common divisor $\leq \min(a, b)$, so there must be a largest.