Equivalence Relation

Recall congruence

$$a \equiv b \pmod{n} \iff n \mid a-b$$

Why is congruence an <u>equivalence relation</u>?

Some numbers in $\mathbb{N}$ become "equivalent" modulo n.

<u>Example</u> : $n=7$

$$\left.\begin{array}{l} \{\ldots, -14, -7, 0, 7, 14, \ldots\} \\ \{\ldots, -13, -6, 1, 8, 15, \ldots\} \\ \{\ldots, -12, -5, 2, 9, 16, \ldots\} \\ \qquad\qquad\vdots \\ \{\ldots, -8, -1, 6, 13, 20, \ldots\} \end{array}\right\} \text{7 classes of equivalence}$$

Given a set S, consider $S \times S$

A relation R is a subset of $S \times S$

$$a R b \iff (a,b) \in R$$

An equivalence relation R (denoted by $\equiv$) satisfies

1. Reflexive. $\forall a \in S, \quad a \equiv a \qquad\qquad (a,a) \in R$
2. Symmetric. $\forall a, b \in S, \quad a \equiv b \iff b \equiv a$
3. Transitive. $\forall a, b, c \in S, \ (a \equiv b \wedge b \equiv c) \Rightarrow a \equiv c$

'$=$' is an equivalence relation.

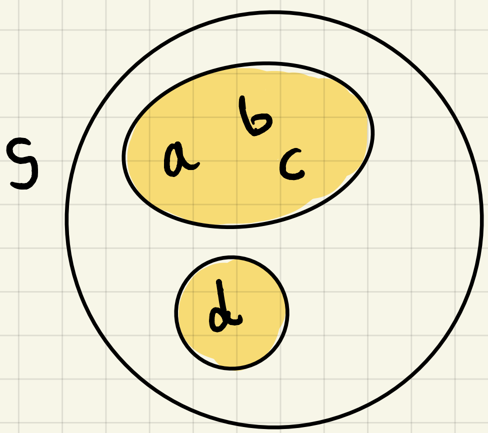
An equivalence relation on S partitions S into classes of equivalence

$$C_a = \{x \in S : a \equiv x\}$$

Example: $S = \{a, b, c, d\}$

$$R = \{(a,a), (b,b), (c,c), (d,d), (a,b), (b,c), (a,c), (b,a), (c,b), (c,a)\}$$

$C_a = \{a, b, c\}$ $\quad$ $C_b = \{a, b, c\}$ $\quad$ $C_c = \{a, b, c\}$ $\quad$ $C_d = \{d\}$



In general:

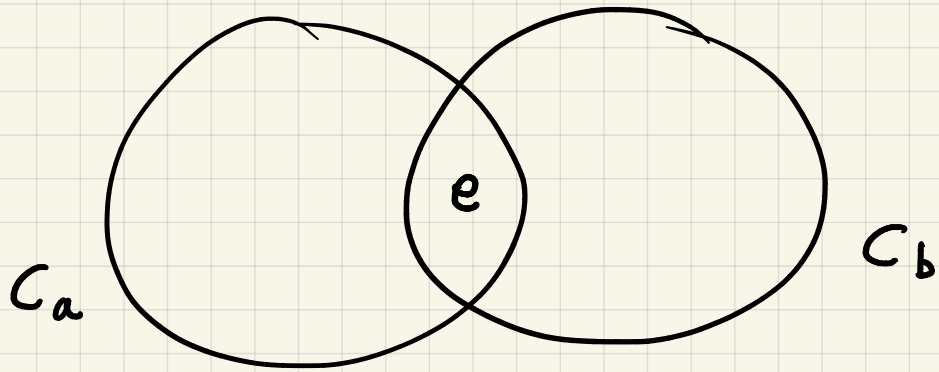
1) $\displaystyle\bigcup_{a \in S} C_a = S$

2) $C_a \cap C_b \neq \emptyset \Rightarrow C_a = C_b$

(either disjoint or the same)

1) By reflexivity: $\forall a \in S$, $a \in C_a$ because $a \equiv a$

Can't have



$e \in C_a \Rightarrow a \equiv e$

$e \in C_b \Rightarrow b \equiv e$

Proof: $x \in C_a \Rightarrow \begin{matrix} a \equiv x \\ a \equiv e \end{matrix} \Rightarrow x \equiv e$

symmetry & transitivity

$\begin{matrix} b \equiv e \\ x \equiv e \end{matrix} \Rightarrow b \equiv x \Rightarrow x \in C_b$. Therefore $C_a \subset C_b$

Similarly, we can show $C_b \subset C_a$. Therefore $C_a = C_b$

Partial order Relation

- Equivalence relation "groups" the elements
- Partial order relation "orders" the elements

Denote a partial order by $\prec$, so $a \prec b$ means $(a,b) \in R$

$\equiv$ to $=$ "is the same as" $\prec$ to $<$

1. Transitive. (as before)
2. Antisymmetric. $\forall a, b \in S$, $(a \prec b \wedge b \prec a) \Rightarrow a = b$
3. $\prec$ could be reflexive or not.

Example: $<$ on $\mathbb{R}$, $\leq$ on $\mathbb{R}$

(not reflexive) (reflexive)

If S is finite, then S admits a minimum

$$\exists e \in S,\ \forall x \in S,\ x \neq e \Rightarrow x \not\prec e$$

<u>proof:</u> Suppose e does not exist, I can find an infinite sequence

$$a_1 \succ a_2 \succ a_3 \cdots\cdots \quad \text{where } a_i \neq a_{i+1}$$

Since S is finite, we must cycle

$$\cdots\cdots \overbrace{a_i \succ \cdots\cdots \succ a_j}^{\text{(transitivity)}} \succ \cdots\cdots \succ a_i \cdots\cdots$$

$$\left.\begin{aligned} a_j &\prec a_i \\ a_i &\prec a_j \end{aligned}\right| \Rightarrow \text{contradiction! (not antisymmetric)}$$

Example: $S = \{a, b, c\}$

$$\mathcal{P}(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

Relation: $X < Y \iff X$ is a proper subset of Y

Transitive. $X < Y \wedge Y < Z \Rightarrow X < Z$

Antisymmetry. $(X < Y \wedge Y > X) \Rightarrow X = Y$

$$\begin{tikzcd}
\text{min} \arrow[rr] & & \phi \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
& \{a\} \arrow[d, dash] \arrow[dr, dash] & \{b\} \arrow[dl, dash] \arrow[dr, dash] & \{c\} \arrow[dl, dash] \arrow[d, dash] \\
& \{a,b\} \arrow[dr, dash] & \{a,c\} \arrow[d, dash] & \{b,c\} \arrow[dl, dash] \\
& & \{a,b,c\} &
\end{tikzcd}$$

"Hasse Diagram."

All edges that can be inferred by transitivity are omitted.

Example: $(a,b) \prec (c,d) \iff (a<c) \vee (a=c \wedge b<d)$

Exercice : Prove this is a partial order relation.

Transitive: $(a,b) \prec (c,d)$

$(c,d) \prec (e,f)$

1) $a<c \quad \wedge \quad c<e \quad \Rightarrow \quad a<e$

2) $a<c \quad \wedge \quad c=e \quad \Rightarrow \quad a<e$

3) $a=c \quad \wedge \quad c<e \quad \Rightarrow \quad a<e$

4) $(a=c \wedge b<d) \wedge (c=e \wedge d<f) \Rightarrow \quad a=e \wedge b<f$

Therefore $(a,b) \prec (e,f)$

$$(a, b) < (c, d) \iff (a < c) \vee (a = c \wedge b < d)$$

Antisymmetry.

$$\left.\begin{array}{l} (a, b) < (c, d) \\ (c, d) < (a, b) \end{array}\right\} \text{can't happen simultaneously}$$

1) $a < c \quad \wedge \quad c < a$ ✗
2) $a < c \quad \wedge \quad c = a$ ✗
3) $a = c \quad \wedge \quad c < a$ ✗
4) $b < d \quad \wedge \quad d < b$ ✗

Note: In general, to prove antisymmetry, prove

either: $x < y \ \wedge \ y < x \implies x = y$

or : $x < y \ \wedge \ y < x$ is false