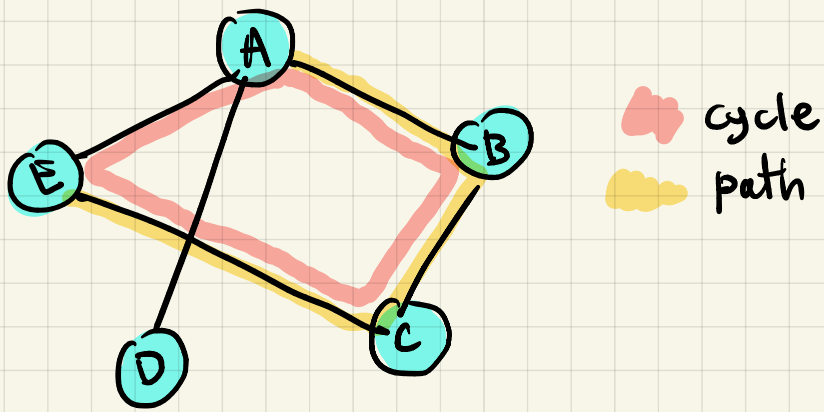
Graph Theory

- A graph consists of a set of vertices V and a set of edges $E \subset V \times V$.
- It represents pairwise relationship, typically illustrated using dots or circles for vertices and lines (not necessarily straight) for edges.

Example:



We will focus on undirected graphs. Edges have no direction.

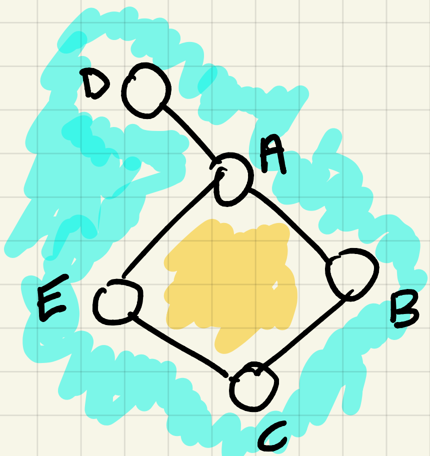
What do we know?

- Degree of a vertex: # edges touching it.
- $\sum_{v \in V} d_v = 2|E| = 2\times \#\text{ edges}$
- Planar graphs: a graph we can draw in the plane without edges crossing

  Euler formula: $v - e + f = 2$

  $v = |V|$, $e = |E|$, $f = \#\text{ faces}$

Example



$\left.\begin{aligned} v &= 5 \\ e &= 5 \\ f &= 2 \end{aligned}\right\}$ verify

A path in the graph is a sequence of vertices connected by edges

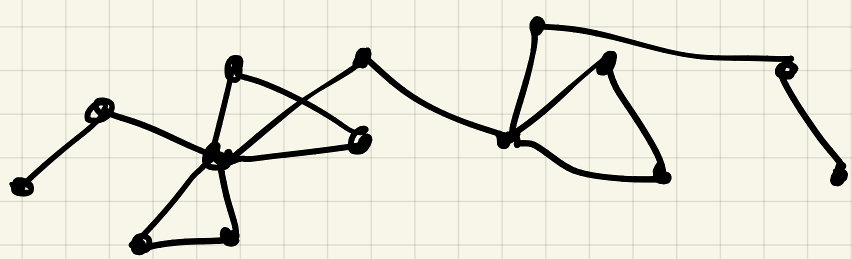
$$v_1, v_2, \ldots, v_n \quad \text{(path from } v_1 \text{ to } v_n\text{)}$$

- $v_i \neq v_j$
- $(v_i, v_{i+1}) \in E$
- All edges are distinct.

Exception: $v_1 = v_n$, path is called a cycle.

Why A B A is not a cycle? (same edge)

A walk is a path where vertices and edges can repeat, and if $v_1 = v_n$, we call it closed walk.

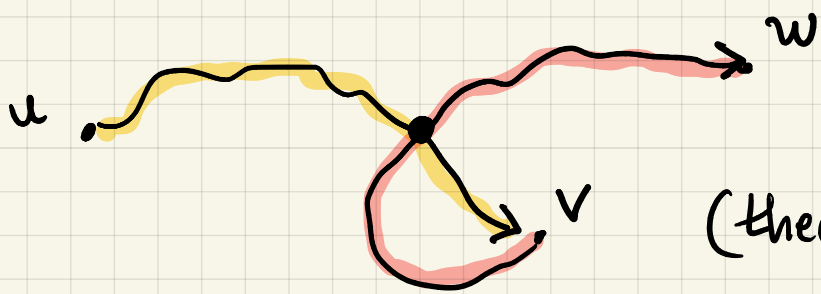


The path relation is an equivalence relation on V

$u \rightsquigarrow v \iff$ there is a path from u to v

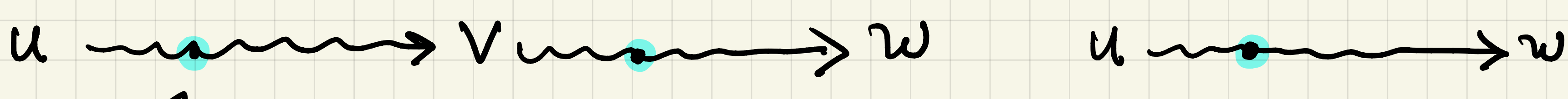
- Reflexive: $u \rightsquigarrow u$ (Empty path)
- Symmetric: $u \rightsquigarrow v \iff v \rightsquigarrow u$ (undirected graph)
- Transitive: $u \rightsquigarrow v \wedge v \rightsquigarrow w \implies u \rightsquigarrow w$

(tricky but true)



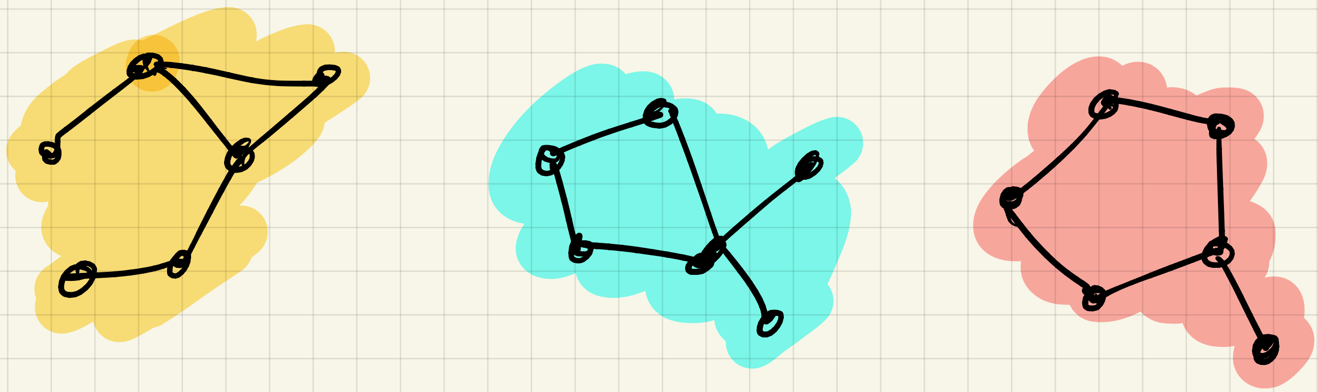
(there is a walk from u to w)

Extrat a path from the walk.



First vertex that repeats

- Every equivalence relation defines classes of equivalence
- What are the classes of equivalence in the graph.
- We call them the <u>connected components</u> of the graph



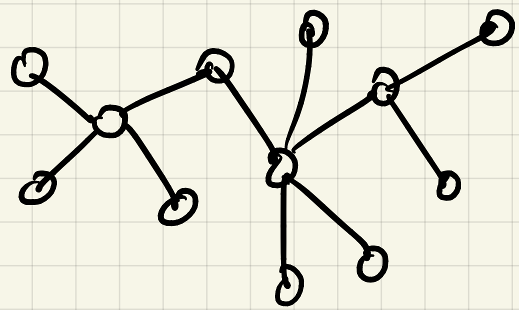
This graph has 3 connected components $c=3$

A graph with $c=1$ is called a connected graph.

Update: Euler formula: $v-e+f=c+1$
(planar graphs)

Trees

A tree is a connected graph with no cycles.



Alternative definitions:

1. connected & cycle-free
2. connected but removing any edge disconnects it.
   [minimally connected]
3. Cycle-free but adding any edge creates a cycle
   [maximally cycle-free]

$$\text{Tree} \Longrightarrow |E| = |V| - 1$$

Tree $\Rightarrow |E| = |V| - 1$

Lemma: Every tree with $|V| \geq 2$ must have at least one vertex with degree 1.

proof: Induction on $n = |V|$

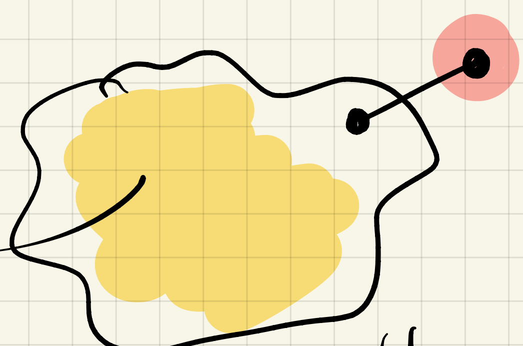
Base case: $n = 1$

$|E| = 0$
$|V| = 1$

$|E| = |V| - 1$

Inductive step: A tree with k vertices has $|E| = |V| - 1$

Consider a tree with $k+1$ vertices, it must have a vertex with degree 1



delete this vertex of degree 1 together with its edge.

a tree with k vertices has $|E| = |V| - 1$

adding back the vertex and its edge maintains the equality.

Proving Euler's formula for $c=1$ (connected graph)

$$v - e + f = 2$$

Induction of # edges.

Base case: $|E| = 0 \quad |V| = 1$

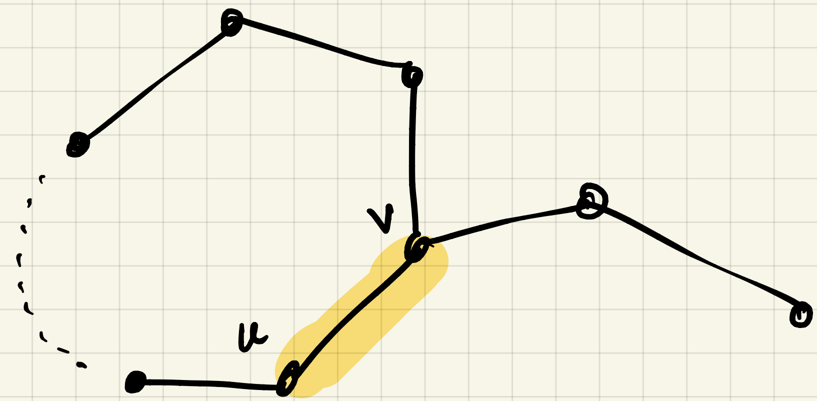
$$v - e + f = 1 - 0 + 1 = 2 \checkmark$$

Inductive step: Assume true for k, consider $k+1$ edges

- If graph is tree, then: $e = k+1$

$$v = k+2$$
$$f = 1$$
$$(k+2) - (k+1) + 1 = 2$$

- If not, then there is a cycle. Delete an edge on the cycle. Now we have a connected graph with k edges, it must have $v - e + f = 2$.

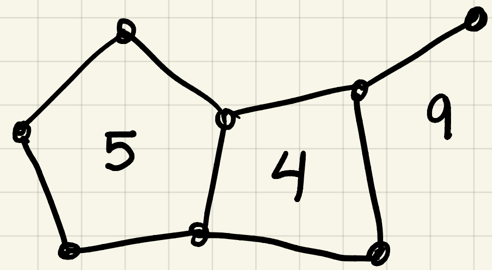


Deleting (u,v) also merges two faces.

So when we add (u,v) back, we get $e+1$ edges and $f+1$ faces, making $v-(e+1)+(f+1)=2$.

Which graphs are not planar ?

First, define degree of a face to be the number of edges on a closed walk of its boundary

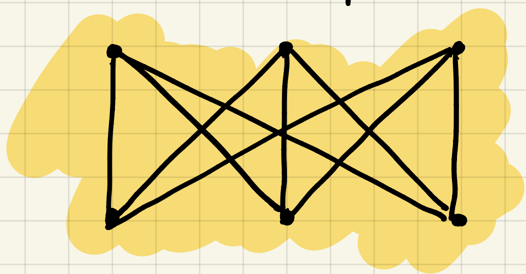


$5+4+9=18$

$e=9$

$$\sum_f d_f = 2e$$

$K_{3,3}$ is not planar

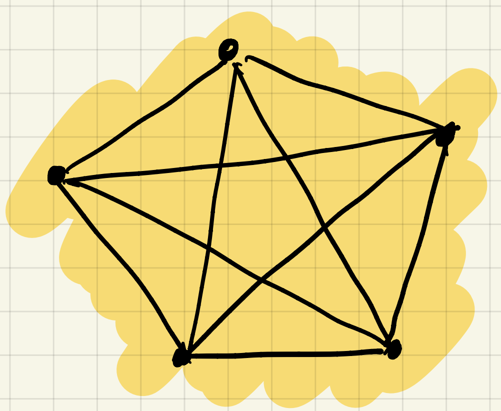


every face has at least 4 edges

$$4f \leq 2e$$

$v = 6,\ e = 9 \Rightarrow f = 5$, but $4 \times 5 > 2 \times 9$

K_5 is not planar



every face has at least 3 edges

$v = 5,\ e = 10 \Rightarrow f = 7$, but $3 \times 7 > 2 \times 10$

Every non-planar graph has $K_{3,3}$ or K_5 as a "basic shape"

Another interesting result:

Every planar graph with $v > 2$ satisfies

$$e \leqslant 3v - 6$$

proof: Since every face has degree at least 3 (because $v > 2$)

we have $3f \leqslant 2e$

but $e = v + f - 2 \leqslant v + \frac{2e}{3} - 2 \Rightarrow e \leqslant 3v - 6$

Average degree

$$\sum_{v \in V} d_v = 2e \leqslant 6v - 12$$

$$\frac{\sum d_v}{v} \leqslant 6 - \frac{12}{v} < 6$$ (there must be a vertex v such that $d_v \leqslant 5$)