Sum & Product notations

upper bound

$$\sum_{i=a}^{b} f(i) = f(a) + f(a+1) + \dots + f(b)$$

lower bound

$$\prod_{i=a}^{b} f(i) = f(a) \times f(a+1) \times \dots \times f(b)$$

Examples:

$$1 + 2 + 3 + \dots + n = \sum_{i=1}^{n} i$$

$$1 \times 2 \times 3 \times \dots \times n = \prod_{i=1}^{n} i$$

Programs to compute Σ and Π

sum

$s \leftarrow 0$

for $i \leftarrow a$ to b

$\quad s \leftarrow s + f(i)$

return s

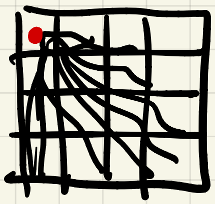
prod

$p \leftarrow 1$

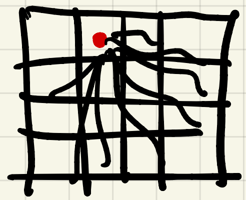
for $i \leftarrow a$ to b

$\quad p \leftarrow p \times f(i)$

return p

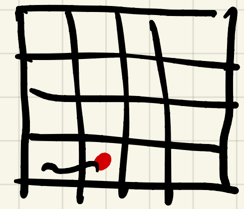


$(n-1)$ snakes



$(n-2)$ snakes

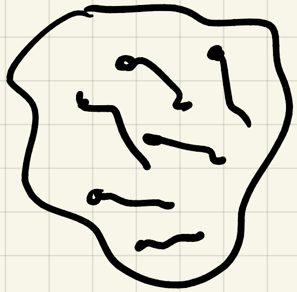
....



1 snakes

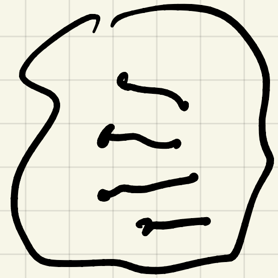
\# possible snakes : $1+2+3+\dots+(n-1) = \dfrac{(n-1)n}{2} = \dbinom{n}{2}$

why I am adding?



S_n

set of snakes with head on square n



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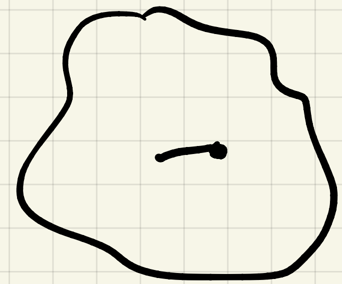


S_2

set of snakes with head on square 2

Notation: $|S|$ = size of set S

size: $|S_n| = n-1$ $\qquad |S_{n-1}| = n-2$ $\qquad |S_2| = 1$

The total # snakes $= |S_n| + |S_{n-1}| + \cdots + |S_2| = \sum_{i=2}^{n} |S_i|$

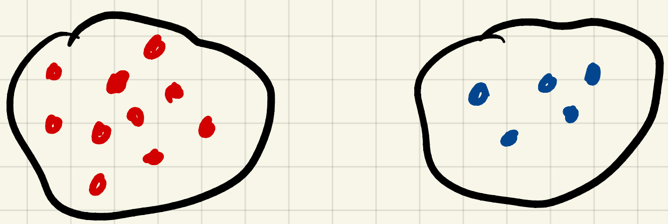
Note: $|S_n| + |S_{n-1}| + \cdots + |S_2| + \underbrace{|S_1|}_{0} = \sum_{i=1}^{n} |S_i|$

Idea above is not always true.

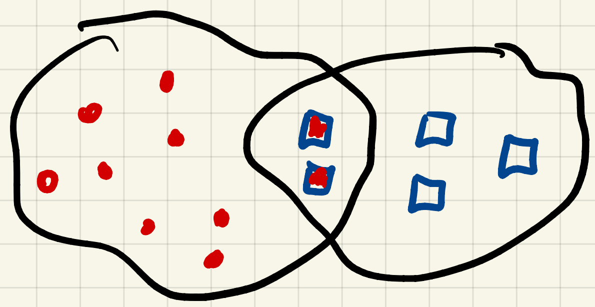
It's true when sets are <u>disjoint</u>

Addition rule: If $S_1, S_2, \ldots, S_k$ are disjoint then the total # elements in their union is $|S_1| + |S_2| + \cdots + |S_k| = \sum_{i=1}^{k} |S_i|$

Example: I have 10 red candies & 5 blue candies so I have 10 + 5 candies in total



I have 10 red candies and 5 square candies, then I don't necessarily have 15 candies



Program that counts all snakes based on the idea of the sum

```
s ← 0
for i ← 1 to n                    (head)
    for j ← 1 to (i-1)            (tail)
        do s ← s + 1              (another snakes)
return s
```

adds $(i-1)$ to s

Math:

$$\sum_{i=1}^{n}\Big(\underbrace{\sum_{j=1}^{i-1} 1}\Big)$$

$$\sum_{i=1}^{n}\big(\underbrace{1+1+1 \cdots 1}_{i-1}\big) = \sum_{i=1}^{n}(i-1)$$

$$= 0+1+2+\cdots+(n-1)$$

$$= n(n-1)/2$$

$$\sum_{i=1}^{n}(i-1) = \sum_{i=1}^{n} i - \sum_{i=1}^{n} 1$$

$$\underbrace{\overbrace{\frac{n(n+1)}{2}}}_{1+2+3+\cdots+n} - n = \frac{n(n+1)-2n}{2} = \frac{n(n-1)}{2}$$

Remember: $\quad n! = 1\times 2\times 3\times \cdots \times n$ (# permutations)

$$\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$1+2+3+\cdots+\square = \frac{\square(\square+1)}{2}$$

Another way of counting snakes

To count something think about a procedure to generate one possibility.

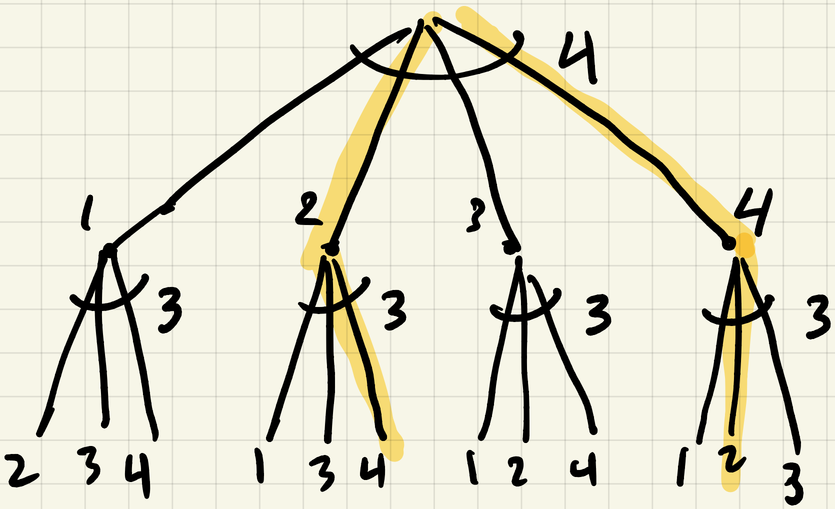
Example:

4	3
1	2

$n=4$

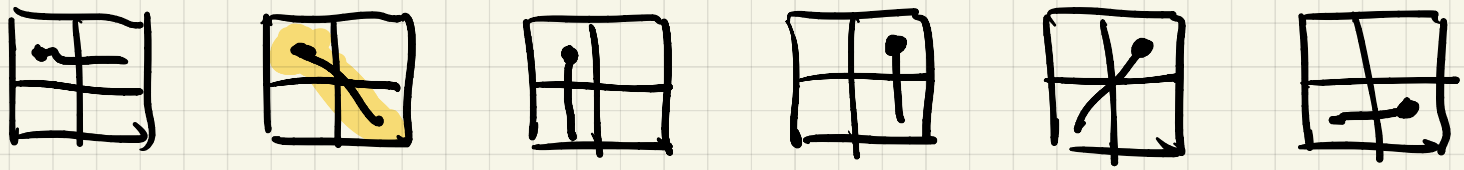
choose a square:

choose another square:



$4 \times 3 = 12$

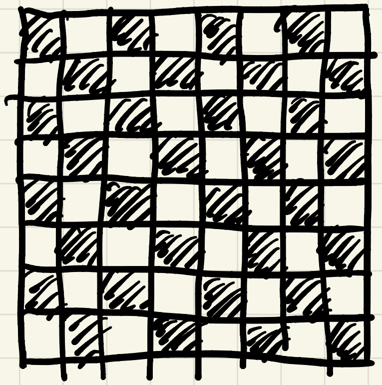
overcount by 2.



Product rule: If a task can be done in k phases, and phase i can be carried out in α_i ways, then the total task can be carried out in $\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_k = \prod_{i=1}^{k} \alpha_i$ ways.

independently from previous phases

$n = 64$



- head & tail must have same color
- head > tail

How many snakes?

	# ways
1. choose a square	64
2. choose a square of the same color	31
	64 x 31

overcount is by 2.

\# snakes: $\frac{64 \times 31}{2}$ $\qquad \frac{n}{2}\left(\frac{n}{2} - 1\right)$

(assuming n even)

In how many ways can I place 2 snakes?

		# ways
Snake 1	1. Choose a square --------	n
	2. Choose another square ------	$n-1$
Snake 2	3. Choose another square ------	$n-2$
	4. Choose " " ------	$n-3$
		$n(n-1)(n-2)(n-3)$

What's the over count? (see below)

ways I can place 2 snakes

$$n(n-1)(n-2)(n-3) / 8$$

$$\Big(\overbrace{\underbrace{\overbrace{a\,,\;b}^{\text{snake 1}}}_{2}\,,\;\underbrace{\overbrace{c\,,\;d}^{\text{snake 2}}}_{2}}^{2}\Big)$$

$$2 \times 2 \times 2 = 8$$