Generalizing permutations and combinations

(1) • There are $n!$ ways of permuting n objects
what if we want to permute some number $k \leq n$
of them? (k-permutation)

(2) • There are $\binom{n}{2} = \frac{n(n-1)}{2}$ ways of choosing pairs
(unordered) out of n objects, what about

$\binom{n}{k}$? ($k \leq n$)

To answer (1), it would help to understand where $n!$ comes from.

Let's generate a permutation:

	# ways
1. Choose the first	$n = n-(1-1)$
2. choose the second	$(n-1) = n-(2-1)$
3. " the third	$(n-2) = n-(3-1)$
⋮	
k. " the k^{th}	$(n-k+1) = n-(k-1)$
⋮	
n. choose the n^{th}	1
	$n(n-1)(n-2)\dots 1 = n!$

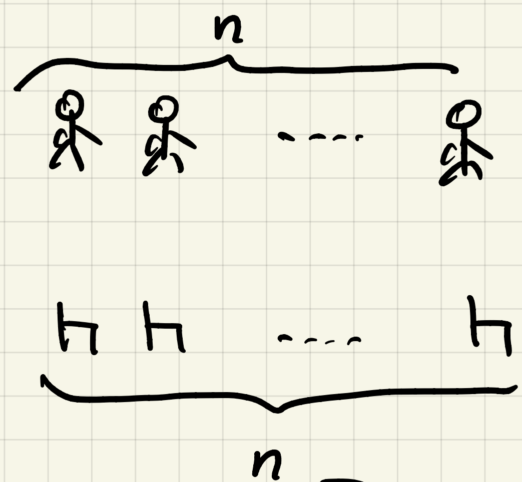
Give above procedure with $\underline{\underline{k}}$ phases, product rule gives the # of k- permutations

$$
{}_nP_k = P^n_k = n(n-1)(n-2) \quad \cdots \quad (n-k+1)
$$

$$
= \prod_{i=0}^{k-1} (n-i) = \prod_{i=1}^{k} (n-i+1)
$$

$$
= \frac{n(n-1)(n-2) \cdots (n-k+1)(n-k)(n-k-1) \cdots 1}{(n-k)(n-k-1) \cdots 1} = \frac{n!}{(n-k)!}
$$

Example:(1)



chairs

- In how many ways can I seat n people on n chairs?

  that's a permutation, so $n!$

- what if we have k chairs $(k \leq n)$

  (not everyone gets to sit)

  $$\# \text{ } k\text{-permutation} = \frac{n!}{(n-k)!}$$

Example(2): We have n movies and k nights $(k \leq n)$.

In how many ways can we decide on what to watch?

Again, that's $\dfrac{n!}{(n-k)!}$

→ To obtain $\binom{n}{k}$ "n choose k" we have to drop order from k-permutations.

→ The k-permutation, over count!

→ By how much? By k!

<u>Example:</u> A, B, C, D

How many 3-permutations are there?

A B C	A B D	A C D	B C D
A C B	A D B	⋮	⋮
B A C	B A D		
B C A	B D A		
C A B	D A B		
C B A	D B A		

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- $$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n!}{2(n-2)!} = \frac{n \times (n-1) \times \cancel{(n-2)!}}{2\cancel{(n-2)!}}$$

$$= \frac{n(n-1)}{2}$$

- $$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n!}{(n-1)!} = \frac{n \times \cancel{(n-1)!}}{\cancel{(n-1)!}} = n$$

- $$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{\cancel{n!}}{1 \cdot \cancel{n!}} = 1.$$

What is $\binom{n}{k}$ really?

It's the number of size-k subsets.

Example: $S = \{a, b, c\}$ $|S| = 3$

size 0 subsets: $\{\ \} = \emptyset$ $\binom{3}{0} = 1$

Size 1 subsets: $\{a\}, \{b\}, \{c\}$ $\binom{3}{1} = 3$

size 2 subsets: $\{a, b\}, \{a, c\}, \{b, c\}$ $\binom{3}{2} = 3$

size 3 subsets: $\{a, b, c\}$ $\binom{3}{3} = 1$

$\#$ subsets $= 1 + 3 + 3 + 1 = 8$ (Addition rule)

$S = \{1, 2, 3, \dots, n\}$ $\quad |S| = n$

subsets (addition rule)

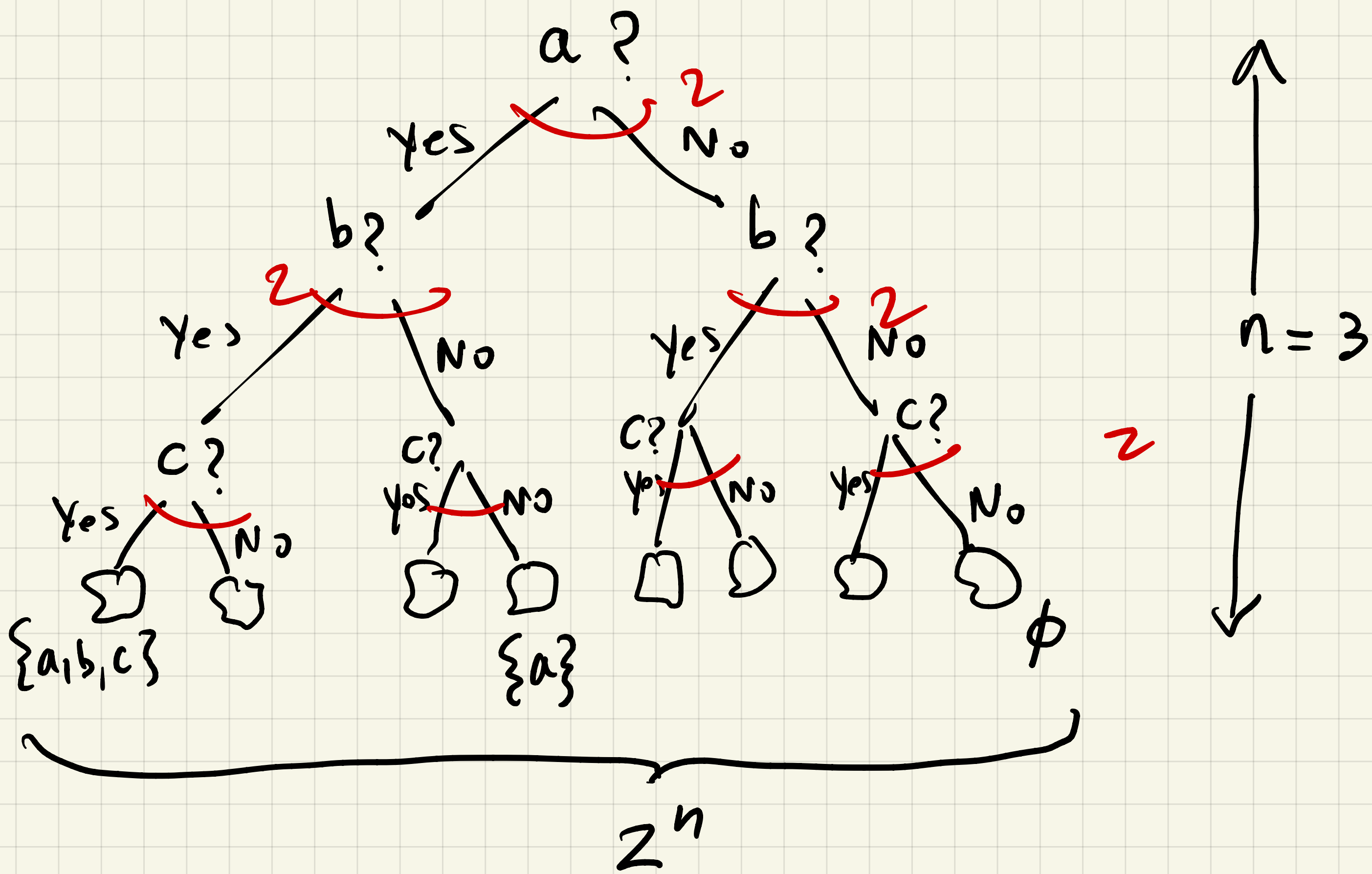
$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} \underbrace{= 2^n}_{\text{later}}$$

Example: $n = 4$

$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 1 + 4 + 6 + 4 + 1 = 16$$

Number of subsets is 2^n

Example: $S = \{a, b, c\}$



		# ways
1.	Choose if a in subset ----	2
2.	choose if b in subset ----	2
	⋮	
		2^n

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Allow repetition in my choice

Example: Make words of length 2 using letters a b c

aa	ba	ca
ab	bb	cb
ac	bc	cc

		# ways
1.	choose a letter ----	3
2.	choose a letter ----	3
		9

In how many ways can we choose k out of n with order & repetition? n^k

	order	no order
no repetition	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$
repetition	n^k	?