Sets and functions

→ A set is an unordered collection of elements

→ We list them, like this

$$S = \{x, y, z\} = \{z, x, y\} \quad , \ |S| = 3$$

→ (Later: we can't always list the element)

Ideas	Notation	Negation
• x is an element of S x is in S	$x \in S$	$x \notin S$
• T is a subset of S every element of T is an element of S	$T \subset S$	$T \not\subset S$

- Empty set $\qquad \phi$, $\{ \}$

Question : Given a set S, is $\phi \subset S$? Yes

- Equality of sets $\qquad$ means $T \subset S$ and $S \subset T$
  $T = S$

- Size of set if S $\qquad |S|$
  is finite

Sets can be tricky

$$S = \{ \underbrace{\{1,2,3\}}_{\text{set}},\ \underbrace{4}_{\text{integer}},\ \underbrace{(5,6)}_{\text{tuple of 2 ints}} \}$$

$1 \notin S$

$\{1,2,3\} \not\subset S$

$\{1,2,3\} \in S$

Some known sets

$$\mathbb{N} = \{1, 2, 3, \ldots\} = \{x \mid x \text{ is a positive integer}\}$$

"such that"

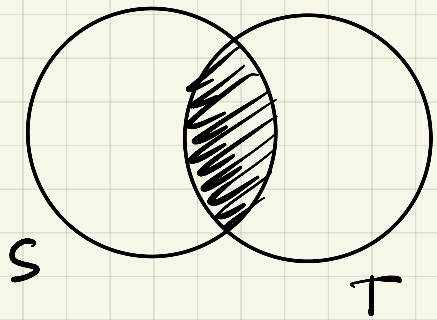
$$\mathbb{Z} = \{\ldots, -3, -2, 1, 0, 1, 2, 3, \ldots\}$$

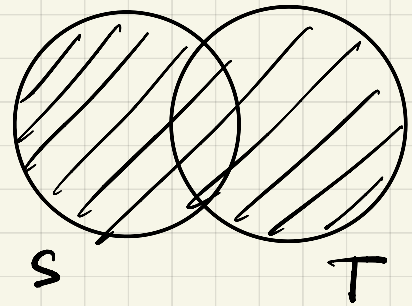
$$\mathbb{Q} = \{x \mid x = \frac{a}{b} \text{ where } a \in \mathbb{Z} \text{ and } b \in \mathbb{N}\}$$

(Rational numbers)

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$$

Intersection & Union





$S \cap T = \{ x \mid x \in S \text{ and } x \in T \}$

$\uparrow$ intersection

$S \cup T = \{ x \mid x \in S \text{ or } x \in T \}$

$\uparrow$ union

Intersection: $S_1 \cap S_2 \cap S_3 \cdots \cap S_n = \bigcap_{i=1}^{n} S_i$

union: $S_1 \cup S_2 \cup S_3 \cdots \cup S_n = \bigcup_{i=1}^{n} S_i$

Addition Rule: $|S \cup T| = |S| + |T|$ if $S \cap T = \emptyset$

Product of sets

$$S \times T = \{(x,y) \mid x \in S \text{ and } y \in T\}$$

Example: $S = \{a, b, c\}$ $T = \{1, 2\}$

$$S \times T = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$$

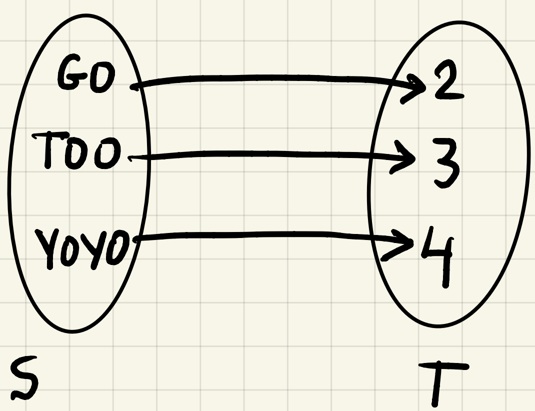
- $S \times T \neq T \times S$
- $|S \times T| = |T \times S| = |S| \cdot |T|$ (why?)

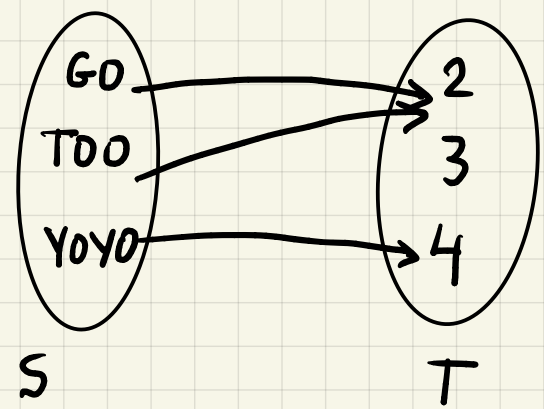
To generate a tuple:

	#ways
1. choose an element from S ...	$\lvert S\rvert$
2. choose an element from T ...	$\lvert T\rvert$
	$\lvert S\rvert \cdot \lvert T\rvert$

Functions

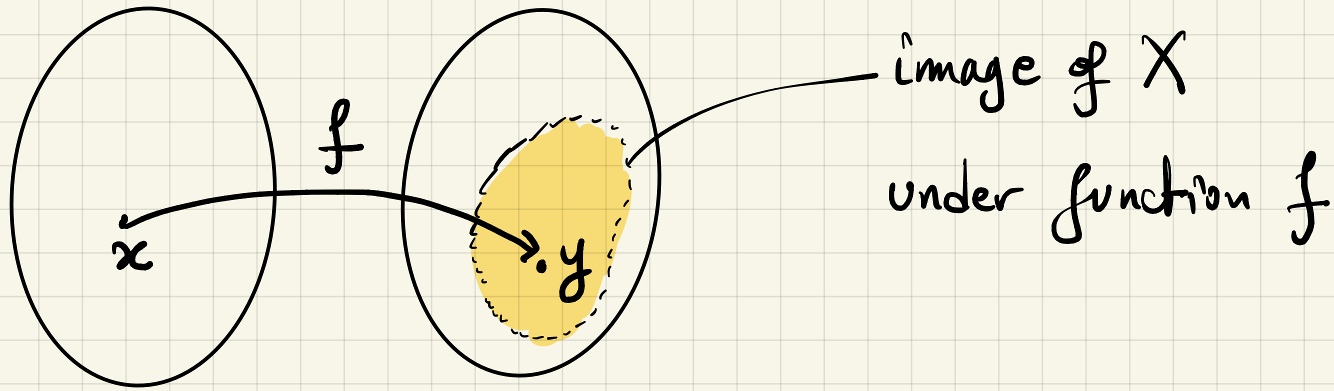
- → A function is a mapping from one set S to another T
- → The function "maps" elements from S to elements in T
- → The function assigns for every element is S exactly one element in T.





Function:

$$f : X \to Y$$



X — Domain — $x \in X$

Y — Co-domain — $y \in Y$

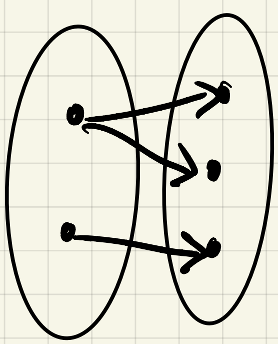
$$y = f(x)$$

When the image is the entire set Y, f is **onto**

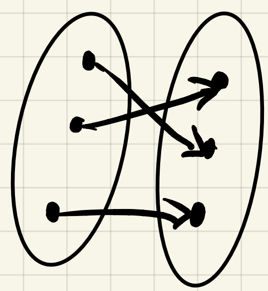
f is a function: For every $x \in X$, there exist exactly one $y \in Y$ such that $y = f(x)$

f is onto: For every $y \in Y$, there exists an $x \in X$ such that $y = f(x)$

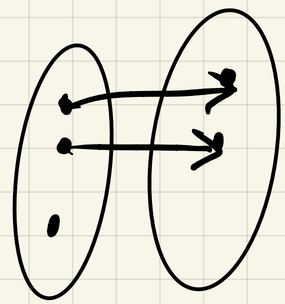
Example: which of the following is a function?



✗



✓ onto



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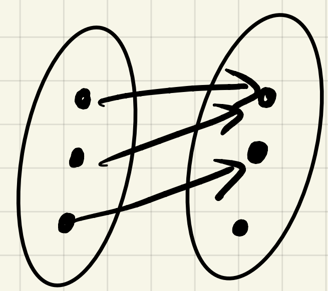


✓ not onto

function : $\forall x \in X$, $\exists$ exactly one y such that $f(x) = y$

$\forall$: universal quantifier "All"

$\exists$: existential quantifier "Exists"

$\exists!$: unique existential quantifier

Function : $\forall x \in X, \exists! y \in Y, f(x) = y$

onto : $\forall y \in Y, \exists x \in X, f(x) = y$

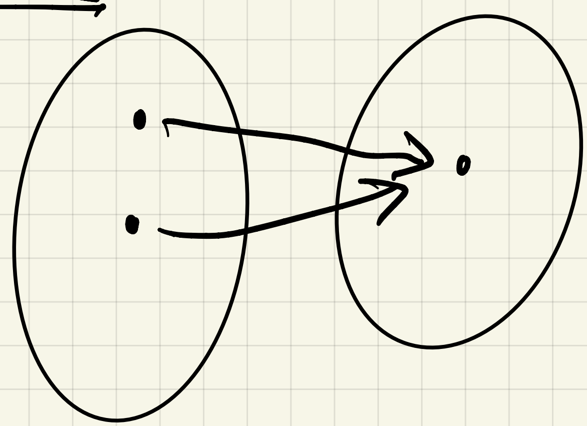
Another property of a function:

one-to-one : For every $x_1, x_2 \in X$,

if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$

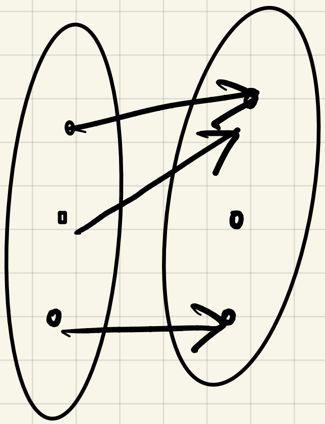
$$\underbrace{\text{if } x_1 \neq x_2 \text{, then } f(x_1) \neq f(x_2)}_{(x_1 \neq x_2) \Longrightarrow f(x_1) \neq f(x_2)}$$

You can't have:



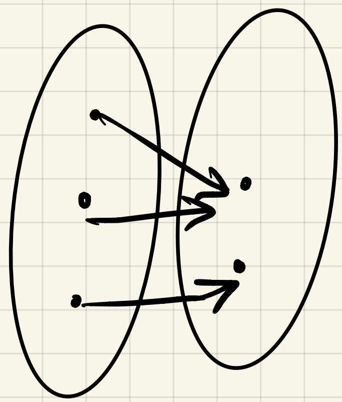
$$\forall x_1 \in X, \forall x_2 \in X, (x_1 \neq x_2) \Longrightarrow f(x_1) \neq f(x_2)$$

$$\forall x_1, x_2 \in X, (x_1 \neq x_2) \Longrightarrow f(x_1) \neq f(x_2)$$



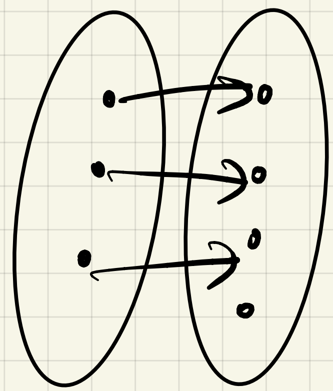
one-to-one: X

onto: X



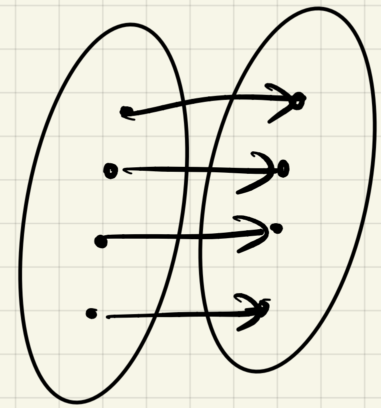
one-to-one: X

onto: ✓



onto-one: ✓

onto: X



One-to-one: ✓

onto: ✓

Bijection

if $f: X \longrightarrow Y$ is a bijection

then $|X| = |Y|$