Examples of functions

	onto	one-to-one	bijection
$f: \mathbb{Z} \to \mathbb{N}$ $f(x) = \lvert x \rvert + 1$	✓	×	×
$g: \mathbb{N} \to \mathbb{Z}$ $g(x) = 2x$	×	✓	×
$h: \mathbb{N} \to \{2, 4, 6, 8 \cdots\}$ $h(x) = 2x$	✓	✓	✓

How do we decide on onto & one-to-one

- How to show that $f: S \to T$ is onto?

  Given $y \in T$, find it's $x \in S$ such that $f(x) = y$

- How to show that $f: S \to T$ is one-to-one

  show if $f(x_1) = f(x_2)$ then $x_1 = x_2$

Example: $f: (0,1) \longrightarrow \mathbb{R}$ $\qquad (0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$

$$f(x) = \ln \frac{x}{1-x}$$

Onto: Given $y \in \mathbb{R}$, how do I find it's $x \in (0,1)$?

It must satisfy $\ln \frac{x}{1-x} = y$

$$\frac{x}{1-x} = e^y$$

$$x = (1-x)e^y \Rightarrow x = e^y - xe^y \Rightarrow x + xe^y = e^y$$

$$x(1+e^y) = e^y$$

$$x = \frac{e^y}{1+e^y} \in (0,1)$$

One-to-one: $f(x_1) = f(x_2)$

$$\ln \frac{x_1}{1-x_1} = \ln \frac{x_2}{1-x_2}$$

$$\frac{x_1}{1-x_1} = \frac{x_2}{1-x_2} \Rightarrow x_1(1-x_2) = x_2(1-x_1)$$

$$x_1 - \cancel{x_1x_2} = x_2 - \cancel{x_2x_1}$$

Binary Patterns

How many binary words can we make with n bits?

	# ways
1. choose a bit ------	2
2. choose a bit ----	2
⋮	⋮
n choose a bit ----	2
	2^n

- A word of length n over the alphabet $\{0,1\}$
  choose n from 2 with order & repetition.

The power set

Example: $S = \{a, b, c\}$

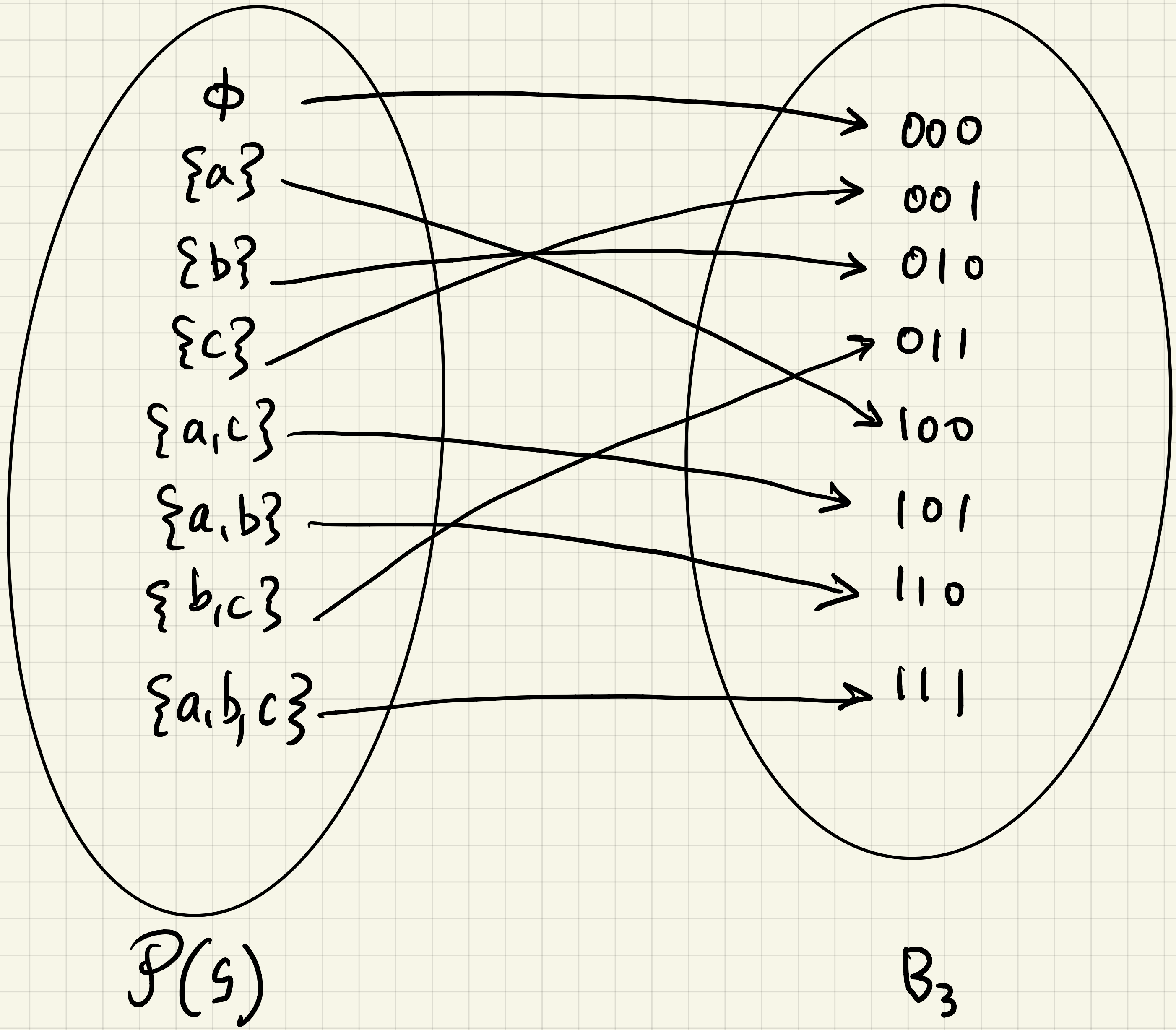
The power set of S

$$\mathcal{P}(S) = 2^S = \left\{ \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \right\}$$

The power set of a set S is the set of all subsets of S

if $|S| = n$, then $|\mathcal{P}(S)| = 2^n$

$f: \mathcal{P}(S) \to B_3$



Let $S = \{1, 2, 3, \dots, n\}$

Given a subset $T \subset S$, let $f(T) = b_1 b_2 \dots b_n$, $b_i \in \{0,1\}$

where $b_i = 1$ if and only if $i \in T$

onto: Given $y = b_1 \dots b_n$, can we find $x = T$ such that

$$f(T) = b_1 \dots b_n$$

Yes: start with $T = \phi$

scan the bits $b_1 \dots b_n$

if $b_i = 1$, $T \leftarrow T \cup \{i\}$

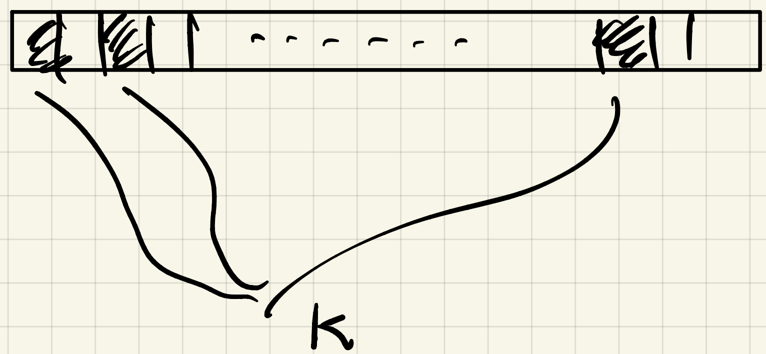
By construction $f(T) = b_1 \dots b_n$

one-to-one: If $f(T_1) = b_1 \dots b_n$, $f(T_2) = w_1 \dots w_n$, $b_1 \dots b_n = w_1 \dots w_n$, then

$T_1 = T_2$ by the above procedure.

Another question about binary words

How many binary words of n bits have exactly k 1s ?



Example: $n = 7$, $k = 3$ $\qquad \binom{7}{3}$

\# ways I can choose 3 bits out of 7 bits.

In general: $\binom{n}{k}$ $\qquad$ [remember this]

Select k out of n

	no order	order	
no rep.	$\binom{n}{k}$	$k!\binom{n}{k} = \frac{n!}{(n-k)!}$	$0 \leq k \leq n$
repetition	?	n^k	$k \geq 0; n \geq 0$

$S = \{a, b, c\}$ $n = 3$

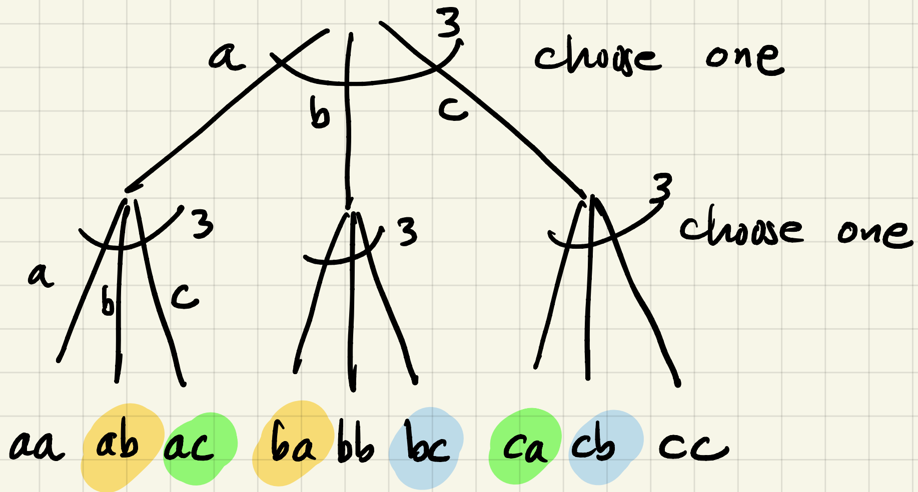
$k = 2$

aa ab ac

ba bb bc

ca cb cc

Product rule does <u>Not</u> work



Some outcomes are overcounted, some are not

can't adjust for overcounting

Different outcomes overcounted differently

We are going to rely on a bijection

$$f$$

aa	$\longrightarrow$	200
ab	$\longrightarrow$	110
ac	$\longrightarrow$	101
bb	$\longrightarrow$	020
bc	$\longrightarrow$	011
cc	$\longrightarrow$	002

set of (x_1, x_2, x_3)
such that $x_1 + x_2 + x_3 = 2$
$\forall i, x_i \geqslant 0$

f is onto

f is one-to-one

$\Downarrow$

f is bijection

$\Downarrow$

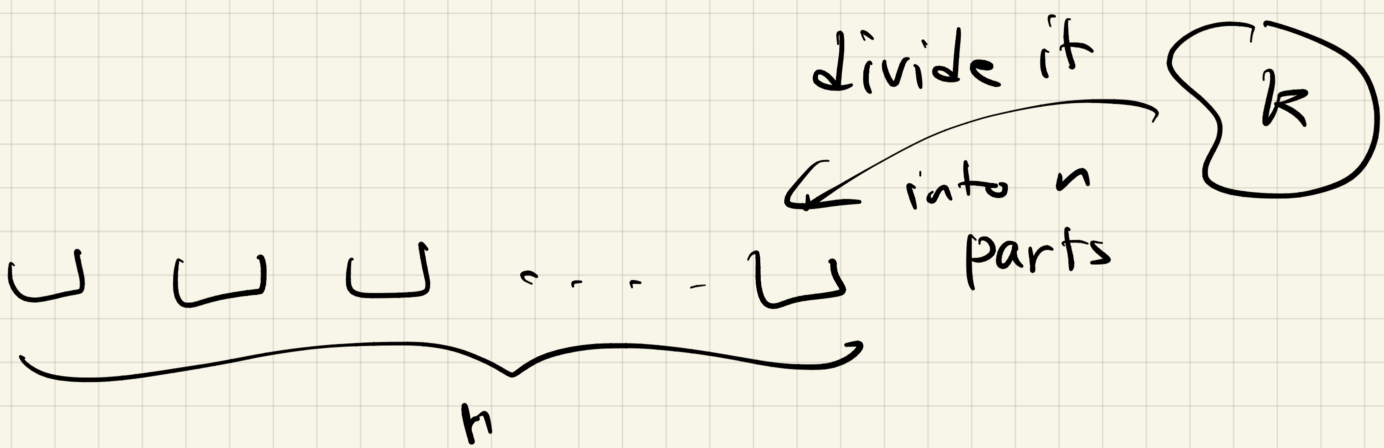
Both sets have same size

To count how many ways we can select k out of n with repetition and no order

We can count instead number of solutions to

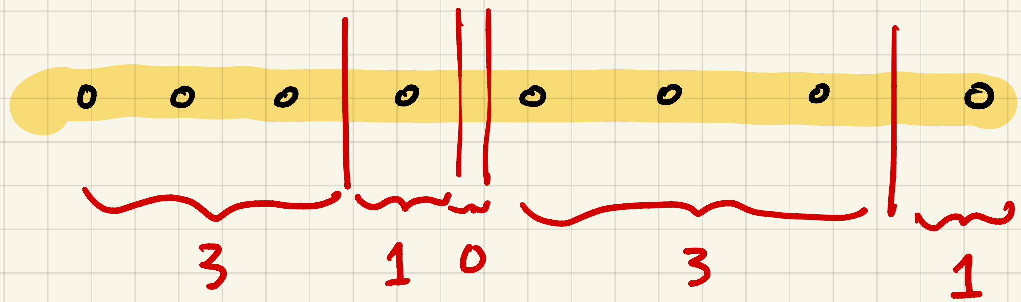
$$x_1 + x_2 + \dots + x_n = k$$

$$x_i \in \{0, 1, 2, 3, \dots\}$$



$K = 8$

$n = 5$



→ In how many ways I can place $(n-1)$ seperators among K things

→ How many binary words have $(n-1)$ 1s and K 0s.

$(n-1)$ 1s and $n-1+K$ bits

→ Ans: $$\binom{n-1+K}{n-1}$$