

Number of integer solutions to

$$x_1 + x_2 + \dots + x_n = k$$

$$x_i \geq 0$$

is $\binom{n+k-1}{n-1}$

Example:

$$x_1 + x_2 + x_3 = 15$$

$$x_i \geq 0$$

$$n=3, k=15 \quad \binom{n+k-1}{n-1} = \binom{3+15-1}{3-1} = \binom{17}{2}$$

What if : $x_1 + x_2 + x_3 = 15$

$$x_1 \geq 0, x_2 \geq 3, x_3 \geq 0$$

Trick : $x_2 = 3 + y_2$, $y_2 \geq 0$

$$x_1 + (3 + y_2) + x_3 = 15$$

$$x_1 + y_2 + x_3 = 12$$

$$x_1 \geq 0, y_2 \geq 0, x_3 \geq 0$$

$$n=3, k=12 : \binom{n+k-1}{n-1} = \binom{3+12-1}{3-1} = \binom{14}{2}$$

Application:

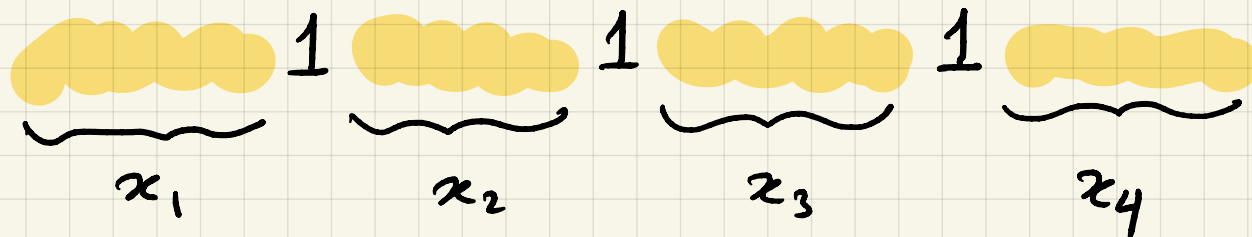
• # of binary words with n bits : 2^n

• # of binary words with n bits & k 1s : $\binom{n}{k}$

→ • # of binary words with n bits & k 1s
& No consecutive 1s ?

Example: $n=10, k=3$

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$$x_1 + x_2 + x_3 + x_4 = 7$$

$$x_1 \geq 0 \quad x_3 \geq 1$$

$$x_2 \geq 1 \quad x_4 \geq 0$$

$$x_2 = 1 + y_2 \quad x_3 = 1 + y_3$$

$$x_1 + 1 + y_2 + 1 + y_3 + x_4 = 7$$

$$x_1 + y_2 + y_3 + x_4 = 5$$

$k=5$
 $n=4$

$$\binom{n+k-1}{n-1} = \binom{5+4-1}{4-1} = \binom{8}{3}$$

Why $\binom{8}{3}$?

Focus on the zeros

. 0 . 0 . 0 . 0 . 0 . 0 . 0 .

- 7 0s define 8 positions.
- Each position can take at most one 1
- Choose 3 out of 8 positions to place the 1s.

That's $\binom{8}{3}$

The binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Some properties:

$$\binom{n}{k} = \binom{n}{n-k} \quad [\text{symmetry}] \quad \text{ex: } \binom{5}{3} = \binom{5}{2}$$

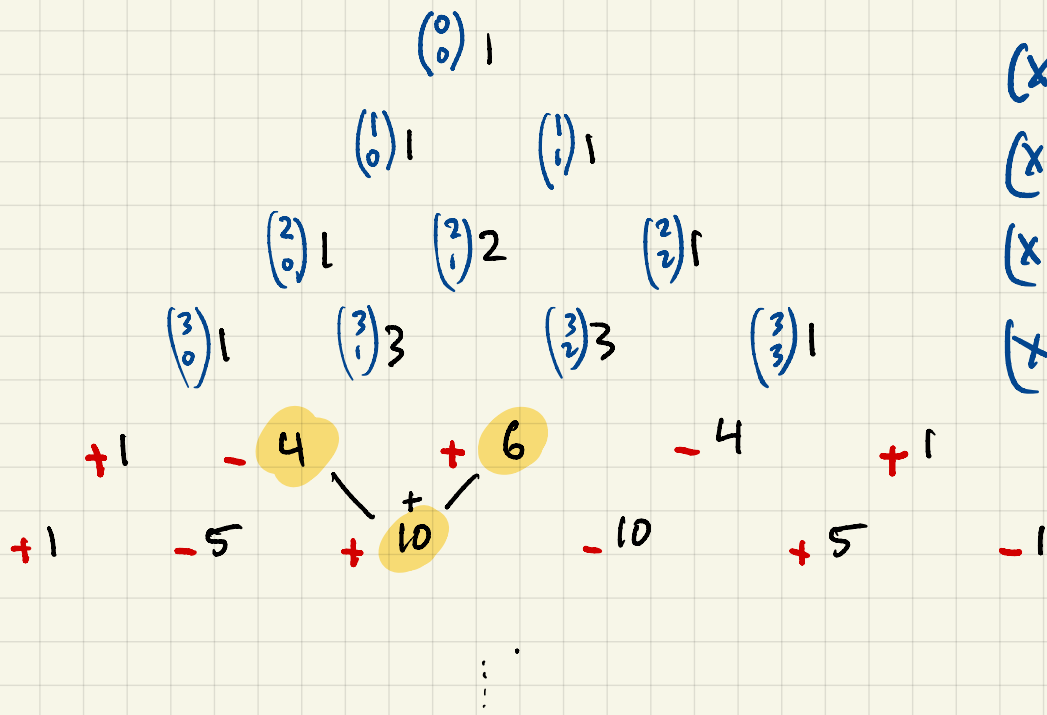
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

[Pascal Triangle]

$$0 < k < n$$

$$\text{ex: } \binom{5}{3} = \binom{4}{2} + \binom{4}{3}$$

Pascal Triangle



$$(x+y)^0 = 1$$

$$(x+y)^1 = 1 \cdot x + 1 \cdot y$$

$$(x+y)^2 = 1 \cdot x^2 + 2 \cdot xy + 1 \cdot y^2$$

$$(x+y)^3 = 1 \cdot x^3 + 3 \cdot x^2y + 3 \cdot xy^2 + 1 \cdot y^3$$

$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}$ are the coefficients of the binomial $(x+y)^n$

Binomial Theorem

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \dots + \binom{n}{n} x^0 y^n$$

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

proof:
$$\underbrace{(x+y)(x+y)\dots(x+y)}_n = \dots \boxed{?} x^{n-k} y^k \dots$$

To generate $x^{n-k} y^k$ we have to pick k factors on the left to contribute y , and that's $\binom{n}{k}$ ways to do it.

Example: $n=3$

$$(x+y)(x+y)(x+y) = \dots \boxed{?} xy^2 \dots$$

$\binom{3}{2} = 3$

Examples

$$\bullet \quad \underset{x \uparrow}{(1+y)}^n = \binom{n}{0} 1^n 1^0 + \binom{n}{1} 1^{n-1} 1^1 + \dots + \binom{n}{n} 1^0 1^n$$

$$= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$$

$$\bullet \quad \underset{x \uparrow}{(1-y)}^n = \binom{n}{0} 1^n (-1)^0 + \binom{n}{1} 1^{n-1} (-1)^1 + \dots + \binom{n}{n} 1^0 (-1)^n$$

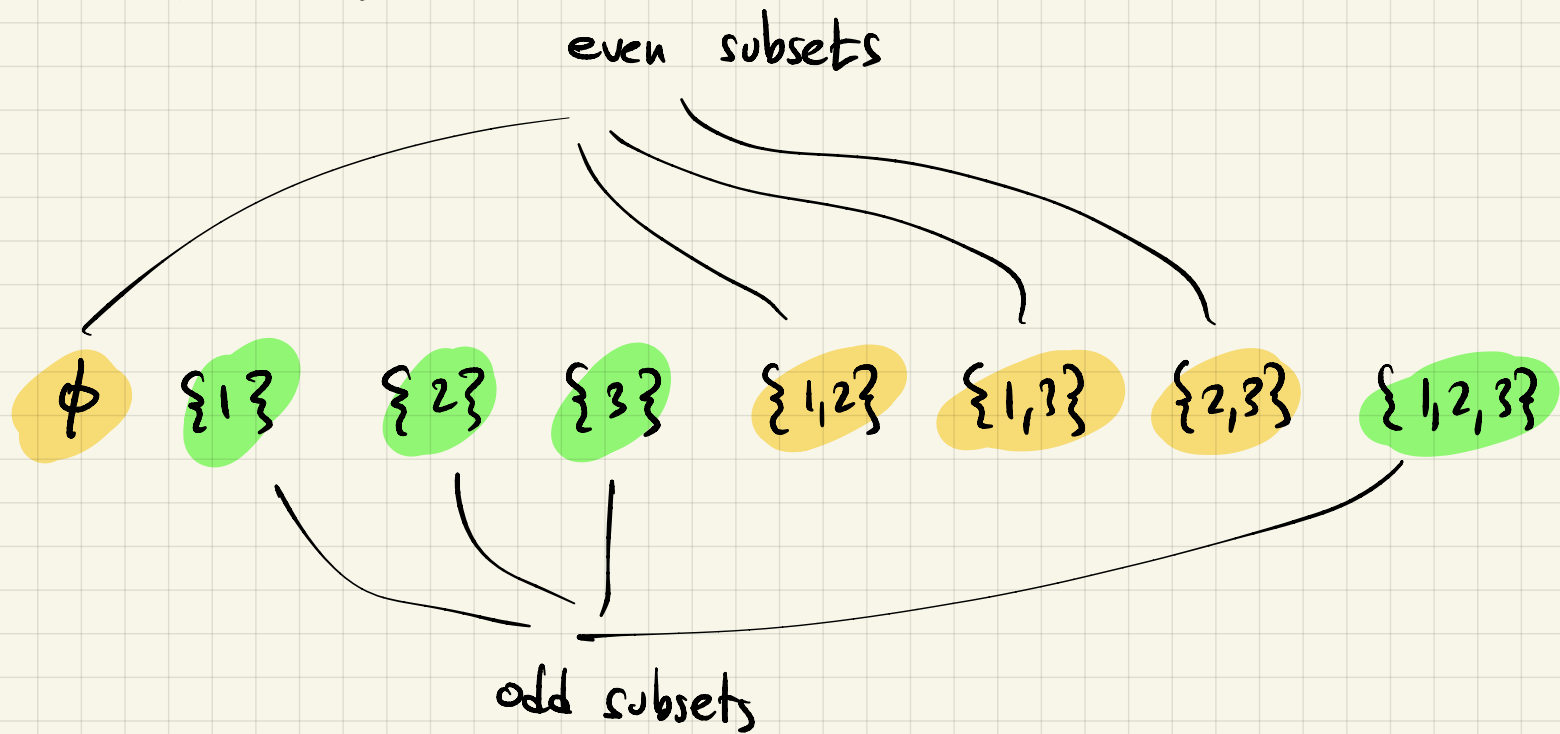
$$= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots + \binom{n}{n} (-1)^n = 0^n$$

$$0^n = \begin{cases} 1 & n=0 \\ 0 & n>0 \end{cases}$$

$$n>0 : \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots$$

For a non-empty set, the # of even subsets = # odd subsets

$$S = \{1, 2, 3\}$$



Generalizing $\binom{n}{k}$

$\binom{n}{k}$ is # binary words with n bits and k 1s.

$$\binom{n}{k} = \frac{n!}{\underbrace{k!}_{\# \text{ 1s}} \underbrace{(n-k)!}_{\# \text{ 0s}}}$$

What if not binary, what if each symbol i occurs k_i times

How many patterns can I make. [anagrams]

$$\# \text{ anagrams} = \frac{(\sum_i k_i)!}{\prod_i (k_i)!}$$

Example: binary $k_1 = k$ $k_0 = n - k$

$$\frac{(k + n - k)!}{k!(n - k)!} = \frac{n!}{k!(n - k)!}$$

Example: How many anagrams can I make from
the words MATHEMATICS.

M: 2	H: 1	C: 1
A: 2	E: 1	S: 1
T: 2	I: 1	

$$\frac{(2 + 2 + 2 + 1 + \dots + 1)!}{2!2!2!1! \dots 1!} = \frac{11!}{2!2!2!}$$