Number of integer solutions to

$$x_1 + x_2 + \cdots + x_n = k$$

$$x_i \geqslant 0$$

is $\binom{n+k-1}{n-1}$

Example:

$$x_1 + x_2 + x_3 = 15$$

$$x_i \geqslant 0$$

$n=3$, $k=15$ $\quad \binom{n+k-1}{n-1} = \binom{3+15-1}{3-1} = \binom{17}{2}$

What if : $x_1 + x_2 + x_3 = 15$

$$x_1 \geqslant 0,\ x_2 \geqslant 3,\ x_3 \geqslant 0$$

Trick : $x_2 = 3 + y_2$, $y_2 \geqslant 0$

$$x_1 + (3 + y_2) + x_3 = 15$$

$$x_1 + y_2 + x_3 = 12$$

$$x_1 \geqslant 0,\ y_2 \geqslant 0,\ x_3 \geqslant 0$$

$n = 3,\ k = 12$: $\binom{n+k-1}{n-1} = \binom{3+12-1}{3-1} = \binom{14}{2}$

Application:

- # of binary words with n bits : 2^n
- # of binary words with n bits & k 1s : $\binom{n}{k}$
- # of binary words with n bits & k 1s & No consecutive 1s ?

Example: $n=10$, $k=3$

0010001010

$$\underbrace{\qquad\quad}_{x_1} 1 \underbrace{\qquad\quad}_{x_2} 1 \underbrace{\qquad\quad}_{x_3} 1 \underbrace{\qquad\quad}_{x_4}$$

$$x_1 + x_2 + x_3 + x_4 = 7$$

$$x_1 \geqslant 0 \qquad x_3 \geqslant 1$$

$$x_2 \geqslant 1 \qquad x_4 \geqslant 0$$

$$x_2 = 1 + y_2 \qquad x_3 = 1 + y_3$$

$$x_1 + 1 + y_2 + 1 + y_3 + x_4 = 7$$

$$x_1 + y_2 + y_3 + x_4 = 5$$

$k=5$
$n=4$

$$\binom{n+k-1}{n-1} = \binom{5+4-1}{4-1} = \binom{8}{3}$$

Why $\binom{8}{3}$? Focus on the zeros

$$. \, 0 \, . \, 0 \, . \, 0 \, . \, 0 \, . \, 0 \, . \, 0 \, . \, 0 \, .$$

- 7 0s define 8 positions.
- Each position can take at most one 1
- Choose 3 out of 8 positions to place the 1s.

That's $\binom{8}{3}$

The binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Some properties:

$$\binom{n}{k} = \binom{n}{n-k} \quad \text{[symmetry]} \qquad \text{ex: } \binom{5}{3} = \binom{5}{2}$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad 0<k<n$$

$$\text{ex: } \binom{5}{3} = \binom{4}{2} + \binom{4}{3}$$

[Pascal triangle]

Pascal Triangle

$\binom{0}{0}\ 1$

$\binom{1}{0}\ 1 \qquad \binom{1}{1}\ 1$

$\binom{2}{0}\ 1 \qquad \binom{2}{1}\ 2 \qquad \binom{2}{2}\ 1$

$\binom{3}{0}\ 1 \qquad \binom{3}{1}\ 3 \qquad \binom{3}{2}\ 3 \qquad \binom{3}{3}\ 1$

$+1 \qquad -4 \qquad +6 \qquad -4 \qquad +1$

$+$

$+1 \qquad -5 \qquad +10 \qquad -10 \qquad +5 \qquad -1$

$\vdots$

$$(x+y)^0 = 1$$

$$(x+y)^1 = 1.x + 1.y$$

$$(x+y)^2 = 1.x^2 + 2.xy + 1y^2$$

$$(x+y)^3 = 1.x^3 + 3.x^2y + 3.xy^2 + 1.y^3$$

$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$ are the coefficient of the binomial $(x+y)^n$

Binomial Theorem

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n} x^0 y^n$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

proof: $$\underbrace{(x+y)(x+y)\cdots(x+y)}_{n} = \cdots\cdots \boxed{?}\, x^{n-k} y^k \cdots\cdots$$

To generate $x^{n-k}y^k$ we have to pick k factors on the left to contribute y, and that's $\binom{n}{k}$ ways to do it.

Example: $n=3$

$$(x+y)(x+y)(x+y) = \cdots \boxed{?}\, xy^2 \cdots$$

$$\binom{3}{2} = 3$$

Examples

- $$\underset{x}{1} + \underset{y}{1}: \quad (1+1)^n = \binom{n}{0} 1^n 1^0 + \binom{n}{1} 1^{n-1} 1^1 + \cdots + \binom{n}{n} 1^0 1^n$$

$$= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

- $$\underset{x}{1} \underset{y}{-1}: \quad (1-1)^n = \binom{n}{0} 1^n (-1)^0 + \binom{n}{1} 1^{n-1} (-1)^1 + \cdots + \binom{n}{n} 1^0 (-1)^n$$

$$= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + \binom{n}{n}(-1)^n = 0^n$$

$$0^n = \begin{cases} 1 & n=0 \\ 0 & n>0 \end{cases}$$

$$n>0: \quad \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

For a non-empty set, the # of even subsets = # odd subsets

$S = \{1, 2, 3\}$

even subsets

$\phi \quad \{1\} \quad \{2\} \quad \{3\} \quad \{1,2\} \quad \{1,3\} \quad \{2,3\} \quad \{1,2,3\}$

odd subsets

Generalizing $\binom{n}{k}$

$\binom{n}{k}$ is # binary words with n bits and k 1s.

$$\binom{n}{k} = \frac{n!}{\underbrace{k!}_{\# 1s}\,\underbrace{(n-k)!}_{\# 0}}$$

What if not binary, what if each symbol i occurs k_i times

How many patterns can I make. [anagrams]

$$\#\text{ anagrams} = \frac{\left(\sum_i k_i\right)!}{\prod_i (k_i)!}$$

Example: binary $\quad k_1 = k \quad k_0 = n-k$

$$\frac{(k+n-k)!}{k!(n-k)!} = \frac{n!}{k!(n-k)!}$$

Example: How many anagrams can I make from the words MATHEMATICS.

M: 2	H: 1	C: 1
A: 2	E: 1	S: 1
T: 2	I: 1	

$$\frac{(2+2+2+1+\cdots+1)!}{2!2!2!1!\cdots1!} = \frac{11!}{2!2!2!}$$