Logic

Recall how to show a function f is one-to-one

if $f(x_1) = f(x_2)$ then $x_1 = x_2$

$$f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$$

if-then

"implies"

Confusing? because of definition of one-to-one?

one-to-one: $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

$f: \mathbb{R} \to \mathbb{R}$

$f(x) = \dfrac{3x+1}{2}$

$$\underbrace{f(x_1) = f(x_2)} \Longrightarrow \frac{3x_1+1}{2} = \frac{3x_2+1}{2}$$

$$\Longrightarrow 3x_1 + 1 = 3x_2 + 1$$

$$\Longrightarrow 3x_1 = 3x_2$$

$$\Longrightarrow \underbrace{x_1 = x_2}$$

Why are the green statements the same?

$$A \Rightarrow B \quad \equiv \quad \text{not } B \Rightarrow \text{not } A$$

Example: My principle in life:

If it's raining, then I carry an umbrella

$(\text{Rain} \Rightarrow \text{Umbrella})$ is true

If Rain is true, what can you claim? Umbrella is true

If Umbrella is true, what can you claim? Nothing

If Umbrella is false, what can you claim? Not Rain

$(\text{not umbrella} \Rightarrow \text{not Rain})$ is true

same

Proofs

- Given a "statement" we want to establish whether it's true or false.
- A statement that is either true or false is a **PROPOSITION**

→ For every non-negative integer n, n^2+n+41 is prime

$$\underset{\text{qualifier}}{\forall} \underbrace{n \in \mathbb{N} \cup \{0\}}_{\text{variable}},\ \underbrace{n^2+n+41 \in \mathbb{P}}_{\text{predicate}}$$ ($\mathbb{P}$ set of primes)

→ There exists an integer greater than zero that is not the product of primes

$$\exists\, n \in \mathbb{N},\ \underbrace{\neg}_{\text{not}}(n \text{ is product of primes})$$

→ For every number x, if $x \geqslant 2$ then $x^2 \geqslant 4$

$$\forall x \in \mathbb{R},\ x \geqslant 2 \Rightarrow x^2 \geqslant 4$$

→ If $a.b$ is irrational, then a is irrational or b is irrational

$$a.b \notin \mathbb{Q} \Longrightarrow a \notin \mathbb{Q} \underbrace{\vee}_{\text{or}} b \notin \mathbb{Q}$$

Let P and Q be propositions:

$P \Rightarrow Q$: P implies Q

$P \vee Q$: P or Q

$P \wedge Q$: P and Q

$\neg P$: not P

$\forall x.\ P(x)$: true if $P(x)$ is true for all x

$\exists x.\ P(x)$: true if $P(x)$ is true for some x

Consider $n^2 + n + 41$

			Prime
$n = 0$	:	$0^2 + 0 + 41 = 41$	✓
$n = 1$	:	$1^2 + 1 + 41 = 43$	✓
$n = 2$	:	$2^2 + 2 + 41 = 47$	✓
$n = 3$	:	$3^2 + 3 + 41 = 53$	✓
	⋮		
$n = 39$	:	$39^2 + 39 + 41 = 1601$	✓
$n = 40$	:	$40^2 + 40 + 41 = 1681$	✗ (counter example)

No proof by example:

- You don't prove that something is always true by giving an example !
- But you can prove $\forall x. P(x)$ is false by providing one counter example x such that $P(x)$ is false

Truth tables

	P	$\neg P$
"false"	0	1
"true"	1	0

Not

P	Q	$P \wedge Q$
0	0	0
0	1	0
1	0	0
1	1	1

AND

P	Q	$P \vee Q$
0	0	0
0	1	1
1	0	1
1	1	1

OR

P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

?

Implication

Why $0 \Rightarrow 0$ is true?

and $0 \Rightarrow 1$ is true?

Consider: $\forall x \in \mathbb{R}, \quad x > 5 \Rightarrow x^2 > 16$

$$\underbrace{x > 5} \Rightarrow \underbrace{x^2 > 16}$$

$x = 4$: $\quad 0 \Rightarrow 0 \quad$ true

$x = 5$: $\quad 0 \Rightarrow 1 \quad$ true

Important observation

$(P \Rightarrow Q)$ is true, this does not say anything about the truth of P or that of Q

P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

Both P and Q can be either true or false

Implication

Other ways of saying $P \Rightarrow Q$

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg Q$	$\neg P \vee Q$	$\neg Q \Rightarrow \neg P$	$P \wedge \neg Q$
0	0	1	1	1	1	1	0
0	1	1	1	0	1	1	0
1	0	0	0	1	0	0	1
1	1	1	0	0	1	1	0

contrapositive

Conclusion: $(P \Rightarrow Q) = (\neg P \vee Q) = (\neg Q \Rightarrow \neg P)$

contrapositive

$$\neg (P \Rightarrow Q) = P \wedge \neg Q$$

Boolean function

$$f : \{0,1\}^n \longrightarrow \{0,1\}$$

$$\{0,1\}^n = \underbrace{\{0,1\} \times \{0,1\} \times \dots \times \{0,1\}}_{n \text{ times}}$$

Example: $f : \{0,1\}^3 \longrightarrow \{0,1\}$

x	y	z	$f(x,y,z)$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Any Boolean function can be constructed using

$\{\neg, \vee, \wedge\}$ operators

We say $\{\neg, \vee, \wedge\}$ is UNIVERSAL

$$f(x,y,z) = (\neg x \wedge \neg y \wedge z) \vee (\neg x \wedge y \wedge \neg z) \vee (x \wedge \neg y \wedge \neg z) \vee (x \wedge y \wedge z)$$