Example functions

	onto	one-to-one	bijection
$f: \mathbb{Z} \to \mathbb{N}$ $f(x) = \lvert x \rvert + 1$	✓	✗	✗
$g: \mathbb{N} \to \mathbb{Z}$ $g(x) = 2x$	✗	✓	✗
$h: \mathbb{N} \to \{2, 4, 6, 8, \ldots\}$ $h(x) = 2x$	✓	✓	✓
$w: \mathbb{N}^2 \to \mathbb{N}$ $w(x, y) = x + y$	✗	✗	✗

Worksheet

	onto	one-to-one	bijection
$f: \mathbb{Z} \to \mathbb{N}$, $f(x) = \lvert x \rvert + 1$	✓ For $y \in \mathbb{N}$, let $x = y-1 \geq 0$; $f(x) = \lvert x \rvert + 1 = x+1 = y-1+1 = y$	✗ $f(-1) = f(1)$	✗
$g: \mathbb{N} \to \mathbb{Z}$, $g(x) = 2x$	✗ No $x \in \mathbb{N}$ such that $f(x) = -1$	✓ $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$	✗
$h: \mathbb{N} \to \{2, 4, 6, 8, \dots\}$, $h(x) = 2x$	✓ For $y \in \{2, 4, \dots\}$, let $x = \frac{y}{2} \in \mathbb{N}$, then $f(x) = 2x = 2\frac{y}{2} = y$.	✓ (same above)	✓
$w: \mathbb{N}^2 \to \mathbb{N}$, $w(x,y) = x+y$	✗ $w(x,y) \geq 2$, No (x,y) such that $w(x,y) = 1$.	✗ $f(x,y) = f(y,x)$	✗

Binary Patterns

- How many binary words with 10 bits can we make?

  2^{10} (why?)

- How many binary words with 10 bits and 3 1s can we make?

  $\binom{10}{3}$ (why?)

- Same as above but consecutive 1s must be separated by at least two 0s.

$$\underbrace{\quad\quad}_{\substack{x_1 \\ \geqslant 0}} 1 \underbrace{\quad\quad}_{\substack{x_2 \\ \geqslant 2}} 1 \underbrace{\quad\quad}_{\substack{x_3 \\ \geqslant 2}} 1 \underbrace{\quad\quad}_{\substack{x_4 \\ \geqslant 0}}$$

$$x_1 + (2 + x_2') + (2 + x_3') + x_4 = 7$$

$$x_1 + x_2' + x_3' + \underbrace{x_4}_{n} = \underbrace{3}_{k}, \quad x_1, x_2', x_3', x_4 \geqslant 0$$

$$\binom{n-k+1}{n-1}$$

$$\binom{4-1+3}{4-1} = \binom{6}{3}$$

The binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\ \dots\ (n-k+1)}{k!}$$

Some properties

[symmetry] $\quad \binom{n}{k} = \binom{n}{n-k} \qquad$ e.g. $\binom{5}{3} = \binom{5}{2}$

[Pascal triangle] $\quad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \quad 0 < k < n$

e.g. $\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$

Algebraic proofs

- $$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!\,k!} = \frac{n!}{(n-k)![n-(n-k)]!} = \binom{n}{n-k}$$

- $$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!}$$

$$= \frac{k(n-1)! + (n-k)(n-1)!}{k!\,(n-k)!}$$

$$= \frac{(n-1)!(k+n-k)}{k!\,(n-k)!} = \frac{(n-1)!\,n}{k!\,(n-k)!} = \frac{n!}{k!(n-k)!}$$

$$= \binom{n}{k}$$

Pascal Triangle

row	
0	1
1	1 1
2	1 2 1
.	1 3 3 1
:	1 4 6 4 1
	1 5 10 10 5 1
	1 6 15 20 15 6 1

$$(x+y)^0 = 1$$
$$(x+y)^1 = 1 \cdot x + 1 \cdot y$$
$$(x+y)^2 = 1 \cdot x^2 + 2 \cdot xy + 1 \cdot y^2$$
$$(x+y)^3 = 1 \cdot x^3 + 3 \cdot x^2 y + 3 \cdot xy^2 + 1 \cdot y^3$$

Let $P(n,k)$ be k^{th} number in row n (both n & k start at 0)

$$P(n,k) = P(n-1,k-1) + P(n-1,k)$$ (does it remind you of something?)

It turns out $P(n,k) = \binom{n}{k}$

- Why are they called Binomial coefficents ?
  - They are the coefficients of $x^{n-k} y^k$ in the expansion of the binomial $(x+y)^n$

- Binomial theorem :

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots \binom{n}{n} y^n$$