Pascal Triangle

row

0 1

1 ..... 1 1

2 .... 1 2 1

⋮ 1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

$$(x+y)^0 = 1$$
$$(x+y)^1 = 1 \cdot x + 1 \cdot y$$
$$(x+y)^2 = 1 \cdot x^2 + 2 \cdot xy + 1 \cdot y^2$$
$$(x+y)^3 = 1 \cdot x^3 + 3 \cdot x^2 y + 3 \cdot xy^2 + 1 \cdot y^3$$

Let $P(n,k)$ be k^{th} number in row n (both n & k start at 0)

$P(n,k) = P(n-1,k-1) + P(n-1,k)$ (does it remind you of something?)

It turns out $P(n,k) = \binom{n}{k}$

- Why are they called Binomial coefficents ?

  ➤ They are the coefficients of $x^{n-k}y^{k}$ in the expansion of the binomial $(x+y)^{n}$

- Binomial theorem :

$$(x+y)^{n} = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k}$$

$$= \binom{n}{0} x^{n} + \binom{n}{1} x^{n-1} y + \dots \binom{n}{n} y^{n}$$

Proof: $\underbrace{(x+y)(x+y)\cdots(x+y)}_{n} = \cdots + \boxed{?}\, x^{n-k}\,y^{k} + \cdots$

To generate $x^{n-k}y^{k}$ we have to pick k out of n factors on the left to contribute y, and there's $\binom{n}{k}$ ways of doing it.

Example: $n=3$

$$(x+y)(x+y)(x+y) = \cdots + \boxed{?}\, xy^2 + \cdots$$

$x \cdot y \cdot y$

$y \cdot x \cdot y$

$y \cdot y \cdot x$

$$\binom{3}{2}$$

Examples :

$$(1+1)^n = \binom{n}{0} 1^n 1^0 + \binom{n}{1} 1^{n-1} 1 + \cdots + \binom{n}{n} 1^0 1^n$$

$$= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n \qquad [\text{familiar?}]$$

$$(1-1)^n = \binom{n}{0} 1^n (-1)^0 + \binom{n}{1} 1^{n-1} (-1)^2 + \cdots + \binom{n}{n} 1^0 (-1)^n$$

$$= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + \binom{n}{n} (-1)^n = 0^n$$

Recall : $0^n = \begin{cases} 1 & n=0 \\ 0 & n>0 \end{cases}$

$n>0$: $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$

$|S| > 0 \Rightarrow$ # even subsets = # odd subsets

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
\times & & +1 & & -4 & & +6 & & -4 & & +1 & & \\
\checkmark & +1 & & -5 & & +10 & & -10 & & +5 & & -1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

The alternating sum is not easily explained by symmetry!

$$S = \{a, b, c\}$$

even subsets

$\binom{3}{0}$ $\binom{3}{2}$

$$\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

$\binom{3}{1}$ $\binom{3}{3}$

odd subsets

Proving it by bijection

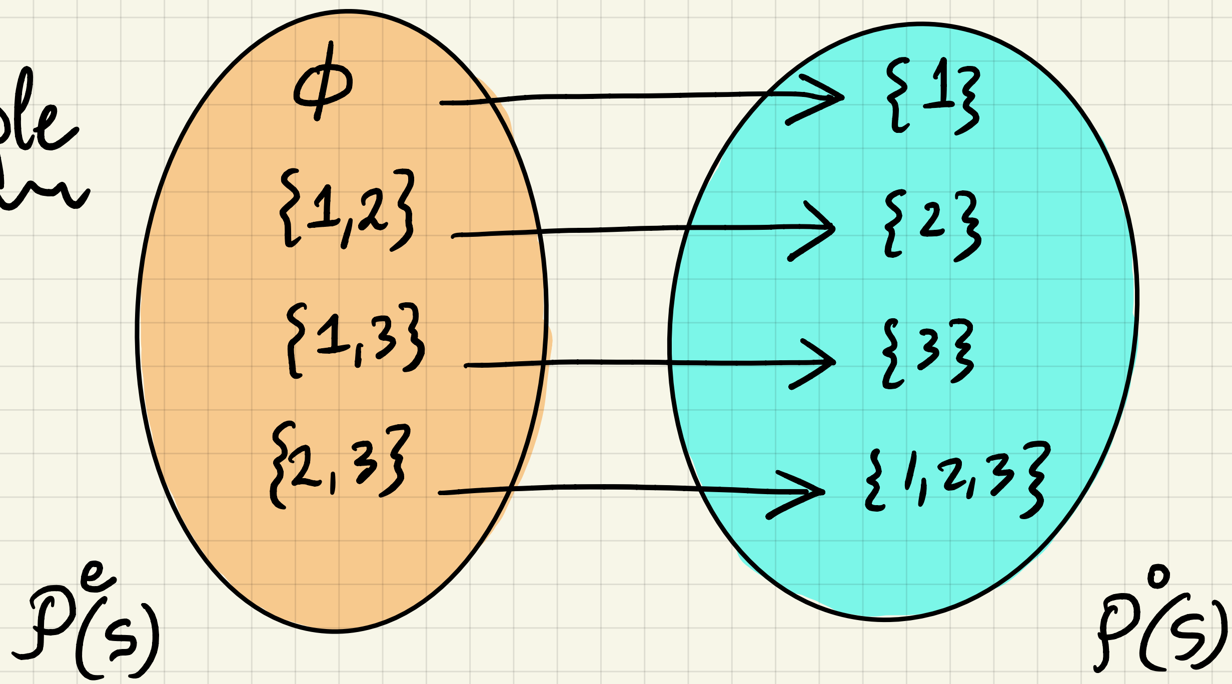
$S = \{1, 2, \dots, n\}$

$f : \mathcal{P}^{e}(S) \longrightarrow \mathcal{P}^{o}(S)$

$$f(x) = \begin{cases} x - \{1\} & 1 \in x \\ x \cup \{1\} & 1 \notin x \end{cases}$$

$1 \in S \Rightarrow$
$|S| > 0$

Example



one-to-one ?

onto ?

onto:

Given $y \in \mathcal{P}(S)$, let

$$x = \begin{cases} y - \{1\} & 1 \in y \\ y \cup \{1\} & 1 \notin y \end{cases}$$

In both case, $|x|$ is even and $f(x) = y$

one-to-one:

Let $f(x_1) = f(x_2) = y$

if $1 \in y$, then $x_1 = y - \{1\}$ and $x_2 = y - \{1\}$, so $x_1 = x_2$

if $1 \notin y$, then $x_1 = y \cup \{1\}$ and $x_2 = y \cup \{1\}$, so $x_1 = x_2$

Generalizing $\binom{n}{k}$

- $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ (binomial coef.)
- $\binom{n}{k_1 \; k_2 \; \dots \; k_m} = \dfrac{n!}{k_1! \, k_2! \dots k_m!}$, $\sum_{i=1}^{m} k_i = n$ (Multinomial coef.)
- So $\binom{n}{k} = \binom{n}{k \quad n-k}$

[Multinomial Theorem]

- $(x_1 + x_2 + \dots + x_m)^n = \displaystyle\sum_{\substack{k_1, \dots, k_m \geq 0 \\ k_1 + \dots + k_m = n}} \binom{n}{k_1 \; \dots \; k_m} x_1^{k_1} \dots x_m^{k_m}$

Anagrams

$$\binom{n}{k_1 \;\ldots\; k_m}$$ counts anagrams!

Examples: How many anagrams of SAAD are there?

$$\binom{4}{1 \;\; 2 \;\; 1} = \frac{4!}{1!\;2!\;1!} = \frac{24}{1\cdot 2\cdot 1} = 12$$

How many anagrams of MATHEMATICS are there?

$$\binom{11}{2\;2\;2\;1\;1\;1\;1\;1} = \frac{11!}{2!\,2!\,2!\,1!\,1!\,1!\,1!\,1!} = 4{,}989{,}600$$

If letter i appears α_i times, then the number of anagrams is

$$\frac{\left(\sum_i \alpha_i\right)!}{\prod_i (\alpha_i!)}$$

factorial of sum

product of factorials

- there are $\left(\sum \alpha_i\right)!$ ways of permuting all letters, but all occurences of letter i can be permuted in $\alpha_i!$ ways, contributing that much overcounting.

Another perspective:

$$\binom{n}{k_1 \;\ldots\; k_m} = \binom{n}{k_1} \cdot \binom{n-k_1}{k_2} \cdot \binom{n-k_1-k_2}{k_3} \cdot \;\ldots\; \cdot \binom{n-k_1-\ldots-k_{m-1}}{k_m}$$

e.g. $$\binom{n}{k_1\; k_2} = \binom{n}{k_1}\binom{n-k_1}{k_2} = \binom{n}{k_1}\binom{k_2}{k_2} = \binom{n}{k_1}$$

Anagram: Choose k_1 positions for letter 1 $\quad \binom{n}{k_1}$

choose k_2 positions for letter 2 $\quad \binom{n-k_1}{k_2}$

$\vdots \qquad\qquad \vdots$

choose k_m positions for letter m $\quad \binom{n-k_1-\ldots-k_{m-1}}{k_m}$

product rule

$$\binom{n}{k_1 \;\ldots\; k_m}$$