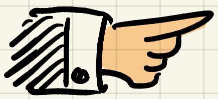
Logic & Proofs

- Recall how we "prove" that a function is one-to-one :

  proof: $\quad f(x_1) = f(x_2) \Rightarrow \dots \Rightarrow x_1 = x_2$

- what is the "logic" behind such proof, especially this was **NOT** how we defined one-to-one !

 : How did we define one-to-one ?

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

Compare to:

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

Do you see the difference ?

- These two are logically equivalent.

$$A \Rightarrow B \qquad\qquad \text{not } B \Rightarrow \text{not } A$$

A real life example

My principle in life:

"If it's raining, then I carry an umbrella"

$$\underbrace{\text{Rain} \Longrightarrow \text{Umbrella}}_{\text{True statement}} \qquad (\text{rain implies umbrella})$$

Rain	Umbrella
True	? True
False	? ✗

Umbrella	Rain
True	? ✗
False	? False

So, $\text{not Umbrella} \Longrightarrow \text{not Rain}$

To understand this better, we need to talk about propositions

Propositision

A proposition is a statement that is either true or false.

A proof establishes the truth value of propositions.

Example Propositions

- For every non-negative integer n, n^2+n+41 is prime

$$\forall \ n \in \mathbb{Z}_{\geq 0}, \ \underline{n^2+n+41 \text{ is prime}}$$

Quantifier   variable   predicate

- There exists a positive integer that is not the product of primes

$$\exists \ n \in \mathbb{N}, \ \neg(n \text{ is a product of primes})$$

NOT

- For every number x, if $x \geqslant 2$ then $x^2 \geqslant 4$

$$\forall x \in \mathbb{R}, (x \geqslant 2 \Rightarrow x^2 \geqslant 4)$$

- If $a \cdot b$ is irrational, then a is irrational or b is irrational

$$a \cdot b \notin \mathbb{Q} \Rightarrow (a \notin \mathbb{Q} \vee b \notin \mathbb{Q})$$

- Let P and Q be propositions. Then the following are propositions

$P \Rightarrow Q$ (P implies Q)
$P \vee Q$ (P or Q)
$P \wedge Q$ (P and Q)
$\neg P$ (not P)

$\forall x, P(x)$: True if $P(x)$ is true for all x

$\exists x, P(x)$: True if $P(x)$ is true for some x

$\forall n \in \mathbb{Z}_{\geq 0}$, n^2+n+41 is prime.

Let's see if above is true.

$n=0$. $0^2+0+41 = 41$ ✓

$n=1$. $1^2+1+41 = 43$ ✓

$n=2$. $2^2+2+41 = 47$ ✓

$n=3$. $3^2+3+41 = 53$ ✓

$\vdots$

$n=39$. $39^2+39+41 = 1601$ ✓

$n=40$. $40^2+40+41 = 1681 = 41^2$

Lesson: No proof by example.

- You don't prove something is always true by giving examples unless you cover all examples
- But you can prove $\forall x, P(x)$ is false by a counter example.
- You can also prove $\exists x, P(x)$ is true by an example.

But how do $\neg, \wedge, \vee, \Rightarrow$ behave?

We can define each by a **Truth Table**.

Truth table: For every possible truth values for P and Q, list the truth value of $\neg P$, $P \wedge Q$, $P \vee Q$, $P \Rightarrow Q$

	P	$\neg P$
"false"	0	1
"true"	1	0

Not

P	Q	$P \wedge Q$
0	0	0
0	1	0
1	0	0
1	1	1

AND

"all"

P	Q	$P \vee Q$
0	0	0
0	1	1
1	0	1
1	1	1

OR

"at least one"

P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

?

Implication

"If-then"

Why are the following true ?

$$0 \Rightarrow 0 \qquad\qquad 0 \Rightarrow 1$$

Consider: $\forall x \in \mathbb{R},\ (x > 5 \Rightarrow x^2 > 16)$

$$\underbrace{x > 5} \Rightarrow \underbrace{x^2 > 16}$$

$x = 4$: $\quad 0 \Rightarrow 0$

$x = 5$: $\quad 0 \Rightarrow 1$

But the statement is true for all x !

Important observation

If $(P \Rightarrow Q)$ is true, this does not say anything about the truth of P or that of Q

P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

Implication

Both P and Q can be either True or False

Other ways of saying $P \Rightarrow Q$

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg Q$	$\neg P \vee Q$	$\neg Q \Rightarrow \neg P$	$P \wedge \neg Q$
0	0	1	1	1	1	1	0
0	1	1	1	0	1	1	0
1	0	0	0	1	0	0	1
1	1	1	0	0	1	1	0

contrapositive

Conclusion: $(P \Rightarrow Q) = (\neg P \vee Q) = (\neg Q \Rightarrow \neg P)$

contrapositive

$$\neg (P \Rightarrow Q) = P \wedge \neg Q$$