Why are the following true?

$0 \Rightarrow 0$ $\qquad\qquad$ $0 \Rightarrow 1$

Consider: $\forall x \in \mathbb{R},\ (x > 5 \Rightarrow x^2 > 16)$

$$\underbrace{x > 5}_{} \Rightarrow \underbrace{x^2 > 16}_{}$$

$x = 4$: $\quad 0 \Rightarrow 0$

$x = 5$: $\quad 0 \Rightarrow 1$

But the statement is true for all x!

Important observation

If $(P \Rightarrow Q)$ is true, this does not say anything about the truth of P or that of Q

P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

Implication

Both P and Q can be either True or False

Other ways of saying $P \Rightarrow Q$

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg Q$	$\neg P \vee Q$	$\neg Q \Rightarrow \neg P$	$P \wedge \neg Q$
0	0	1	1	1	1	1	0
0	1	1	1	0	1	1	0
1	0	0	0	1	0	0	1
1	1	1	0	0	1	1	0

contrapositive

Conclusion: $(P \Rightarrow Q) = (\neg P \vee Q) = (\neg Q \Rightarrow \neg P)$

contrapositive

$$\neg (P \Rightarrow Q) = P \wedge \neg Q$$

"if and only if"

If and only if (iff)

Example: n is even if and only if n^2 is even

$$n \text{ is even} \Leftrightarrow n^2 \text{ is even}$$

$P \Leftrightarrow Q$ is true means:

- $P \Rightarrow Q$ is true
- $Q \Rightarrow P$ is true

observe that we often omit "is true" when we say "P implies Q"

Compare truth tables

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
0	0	1	1
0	1	1	0
1	0	0	1
1	1	1	1

P	Q	$P \Leftrightarrow Q$
0	0	1
0	1	0
1	0	0
1	1	1

Boolean function

$$f : \{0,1\}^n \longrightarrow \{0,1\}$$

$$\{0,1\}^n = \underbrace{\{0,1\} \times \{0,1\} \times \dots \times \{0,1\}}_{n \text{ times}}$$

Example: $f : \{0,1\}^3 \longrightarrow \{0,1\}$

x	y	z	$f(x,y,z)$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

2^3

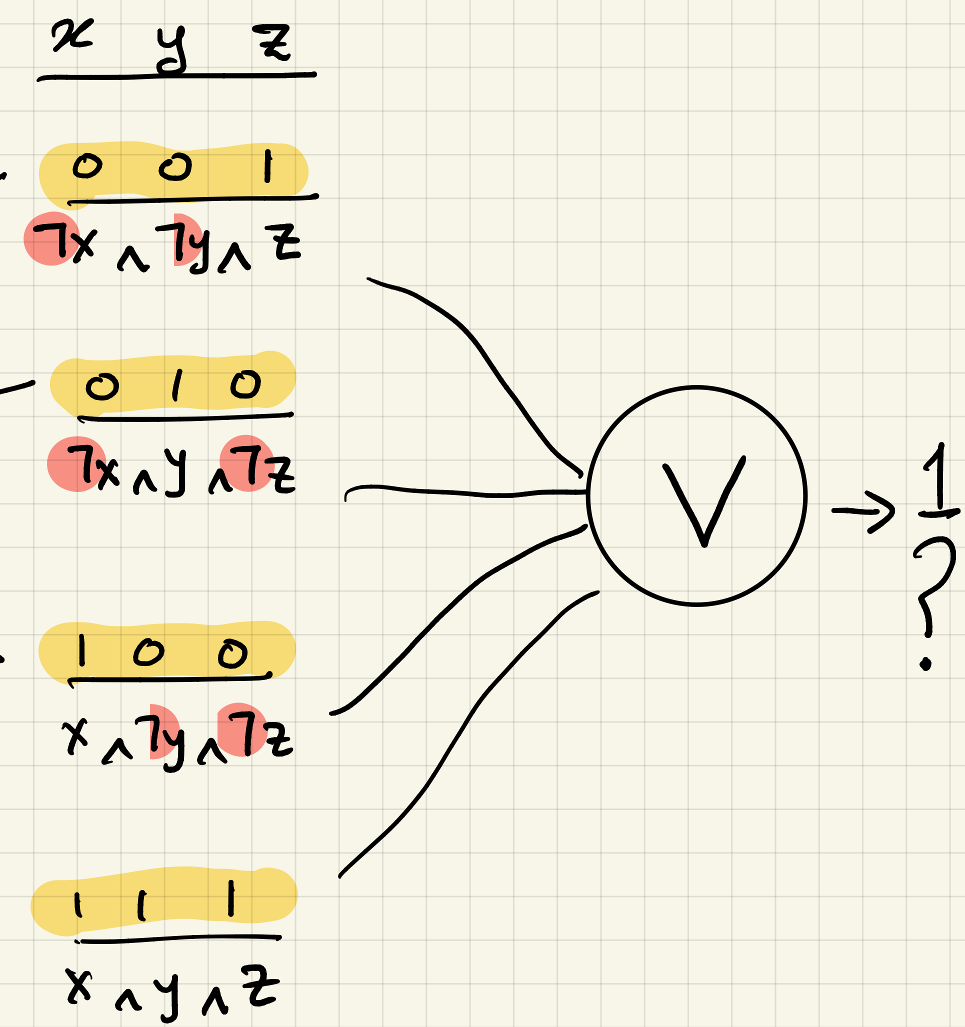
Any Boolean function can be constructed using $\{\neg, \vee, \wedge\}$ operators

We say $\{\neg, \vee, \wedge\}$ is UNIVERSAL

$$f(x,y,z) = (\neg x \wedge \neg y \wedge z) \vee (\neg x \wedge y \wedge \neg z) \vee (x \wedge \neg y \wedge \neg z) \vee (x \wedge y \wedge z)$$

(See below)

x	y	z	$f(x,y,z)$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1



$\{\neg, \wedge\}$ is universal : $\vee$ can be replaced by $\neg, \wedge$

$\{\neg, \vee\}$ is universal : $\wedge$ can be replaced by $\neg, \vee$

De Morgan's Law:

$$\left.\begin{aligned} \neg(A \wedge B) &= \neg A \vee \neg B \\ \neg(A \vee B) &= \neg A \wedge \neg B \end{aligned}\right\} \text{proof? } \underline{\text{truth table}}$$

Example: $\overbrace{a \notin \mathbb{Q}}^{A} \vee \overbrace{b \notin \mathbb{Q}}^{B}$

Negate above statement : $\underbrace{a \in \mathbb{Q}}_{\neg A} \wedge \underbrace{b \in \mathbb{Q}}_{\neg B}$

Commutative:

$$A \wedge B = B \wedge A$$

$$A \vee B = B \vee A$$

Associative:

$$A \wedge (B \wedge C) = (A \wedge B) \wedge C = A \wedge B \wedge C$$

$$A \vee (B \vee C) = (A \vee B) \vee C = A \vee B \vee C$$

Distributive:

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$$