Properties of implication

$P\,Q$	$P \Rightarrow Q$
0 0	1
0 1	1
1 0	0
1 1	1

① P is true and $(P \Rightarrow Q)$ is true, then Q is true

To prove Q knowing P is true, prove $(P \Rightarrow Q)$ is true

Direct proof

② $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$

To prove $(P \Rightarrow Q)$ is true, prove $(\neg Q \Rightarrow \neg P)$ is true instead

proof by contrapositive

In general, to prove $(P \Rightarrow Q)$ is true,
prove that whenever P is true, Q is also true.
Note: we often omit "is true" when we say "P implies Q".

③ $(\neg P \Rightarrow \text{False})$, then P is true

To prove P is true, prove that $\neg P$ implies something False

proof by contradiction

④ $(P \Rightarrow Q)$ is true and $(Q \Rightarrow R)$ is true,
then $(P \Rightarrow R)$ is true

Chain your implications

transitivity

Let's prove property ④

By case analysis of P

P false : then $(P \Rightarrow R)$ is true regardless of R

P true : P is true and $(P \Rightarrow Q)$ is true,

then Q is true. (property ①)

Now, Q is true and $(Q \Rightarrow R)$ is true,

then R is true (property ①)

P is true and R is true, then

$(P \Rightarrow R)$ is true

Prove that if $x \geq 3$, then $x^2 \geq 9$

$$\underbrace{x \geq 3}_{P} \Rightarrow \underbrace{x^2 \geq 9}_{Q}$$

Prove $(P \Rightarrow Q)$ is true

$$x \geq 3 \underset{\text{algebra}}{\Longrightarrow} x \cdot x \geq 3 \cdot 3 \underset{\text{algebra}}{\Longrightarrow} x^2 \geq 9$$

We proved that whenever $x \geq 3$ is true, $x^2 \geq 9$ is true

We effectively proved : $\forall x \in \mathbb{R},\ x \geq 3 \Rightarrow x^2 \geq 9$

Question: Can we prove $\forall x, y \in \mathbb{R},\ \frac{x}{y} > 1 \Rightarrow x > y$?

$$\frac{x}{y} > 1 \underset{\text{algebra ?}}{\Longrightarrow} x > y \qquad \text{(multiply by } y \text{ on both sides)}$$

$\underbrace{n \text{ odd}}_{P} \Longrightarrow \underbrace{n^2 \text{ is odd}}_{Q}$ (is true)

Definition:

n odd $\Leftrightarrow n = 2 \cdot k + 1,\ k \in \mathbb{Z}$

n even $\Leftrightarrow n = 2 \cdot k,\ k \in \mathbb{Z}$

Direct proof

n is odd $\xRightarrow[\text{def.}]{} n = 2k+1,\ k \in \mathbb{Z}$

$n = 2k+1 \xRightarrow[\text{algebra}]{} n^2 = (2k+1)^2 = 4k^2 + 4k + 1$

$= 2\underbrace{(2k^2 + 2k)}_{k' \in \mathbb{Z}} + 1$

$n^2 = 2k' + 1 \xRightarrow[\text{def.}]{}$ n^2 is odd.

We also proved: n^2 is even $\Longrightarrow$ n is even. (why?)

Example 2: Prove $\underbrace{ab \notin \mathbb{Q}}_{A} \Longrightarrow \underbrace{(a \notin \mathbb{Q} \vee b \notin \mathbb{Q})}_{B}$

Consider the contrapositive:

$$\neg B \Rightarrow \neg A$$

contrapositive

$$a \in \mathbb{Q} \wedge b \in \mathbb{Q} \Longrightarrow ab \in \mathbb{Q}$$

- $a \in \mathbb{Q} \Rightarrow a = \frac{x}{y}$, $x \in \mathbb{Z}$ and $y \in \mathbb{N}$
- $b \in \mathbb{Q} \Rightarrow b = \frac{z}{w}$, $z \in \mathbb{Z}$ and $w \in \mathbb{N}$

$$a \in \mathbb{Q} \wedge b \in \mathbb{Q} \Longrightarrow a = \frac{x}{y} \text{ and } b = \frac{z}{w}$$

$$\begin{matrix} P \Rightarrow Q \\ W \Rightarrow R \end{matrix} \vdash P \wedge W \Rightarrow Q \wedge R$$

(why?) see below

$$a = \frac{x}{y} \quad \text{and} \quad b = \frac{z}{w} \implies$$

$$ab = \frac{x}{y} \cdot \frac{z}{w} = \frac{x.z}{y.w}, \qquad x.z \in \mathbb{Z},\ y.w \in \mathbb{N}$$

$$\implies \underline{ab \in \mathbb{Q}}$$

Remark : If $P \wedge W$ is true, then P is true and W is true. Since $(P \Rightarrow Q)$ is true, then Q is true. Similarly, since $(W \Rightarrow R)$ is true, then R is true. Therefore $Q \wedge R$ is true . So $P \wedge W \Rightarrow Q \wedge R$

Example 3: Prove $\sqrt{2}$ is irrational.

by contradiction

$P: \sqrt{2} \notin \mathbb{Q}$

$\neg P: \sqrt{2} \in \mathbb{Q}$

$\sqrt{2} \in \mathbb{Q} \Rightarrow \sqrt{2} = \frac{a}{b}$ (a & b are integers and $\frac{a}{b}$ is irreducible)

$\Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2b^2 \Rightarrow a^2$ is even $\Rightarrow$ a is even.

$$\sqrt{2} \in \mathbb{Q} \Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow b^2 = \frac{a^2}{2} = \frac{2k \times 2k}{2} = \frac{4k^2}{2} = 2k^2$$

$\Rightarrow b^2$ is even $\Rightarrow$ b is even

a is even and b is even $\Rightarrow \frac{a}{b}$ is reducible

$$\sqrt{2} \in \mathbb{Q} \Rightarrow \underbrace{\left(\frac{a}{b} \text{ is reducible and } \frac{a}{b} \text{ is irreducible}\right)}_{\text{False}}$$