

## Generalizing even/odd results

Prove: (1) •  $n$  even  $\implies n^k$  even,  $k \in \mathbb{N}$

(2) •  $n$  odd  $\implies n^k$  odd,  $k \in \mathbb{N} \cup \{0\}$

(1):  $n$  is even  $\implies n = 2m$ ,  $m \in \mathbb{Z}$

$$\implies n^k = (2m)^k = 2^k \cdot m^k = 2 \underbrace{\left(2^{k-1} \cdot m^k\right)}_{\in \mathbb{Z} ?}$$

$$= 2 \cdot m', \quad m' \in \mathbb{Z}$$

$\implies n^k$  is even.

yes

$$(2) : \quad n \text{ is odd} \Rightarrow n = (2m + 1), \quad m \in \mathbb{Z}$$

$$\Rightarrow n^k = (2m + 1)^k$$

$$= \binom{k}{0} (2m)^k + \binom{k}{1} (2m)^{k-1} + \dots + \binom{k}{k-1} (2m)^1 + \binom{k}{k} (2m)^0$$

$$\underbrace{\hspace{15em}}$$

$$2 \left[ \binom{k}{0} 2^{k-1} m^k + \dots + \binom{k}{k-1} 2^0 m \right] + 1$$

$$\in \mathbb{Z}$$

$$\Rightarrow n^k \text{ is odd}$$

empty sum when  $k=0$

Conclusion:  $n$  is even  $\iff n^k$  is even,  $k \in \mathbb{N}$

$n$  is odd  $\iff n^k$  is odd,  $k \in \mathbb{N}$

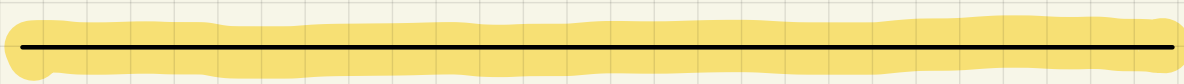
In other words,  $n, n^2, n^3, \dots$  have the same parity!

How do we prove iff

$$P \Leftrightarrow Q$$

1) Prove  $P \Rightarrow Q$  is true

2) Prove  $Q \Rightarrow P$  is true



1) Prove  $P \Rightarrow Q$  is true

2) Prove  $\neg P \Rightarrow \neg Q$  is true

Contrapositive

Prove: There is no smallest positive rational number.

By contradiction:

Let  $x > 0$  be the smallest positive rational number

let  $y = \frac{x}{2}$  ( $y > 0$ ,  $y \in \mathbb{Q}$  because if  $x = \frac{a}{b}$  then  $y = \frac{a}{2b}$ )

contradiction since  $y < x$ .

Prove :  $n$  odd  $\Rightarrow n = a^2 - b^2$  where  $a, b \in \mathbb{Z}$

$$n \text{ is odd} \Rightarrow n = 2k + 1, k \in \mathbb{Z}$$

$$= \underbrace{(k+1)^2}_{k^2 + 2k + 1} - k^2$$

contrapositive:  $n \neq a^2 - b^2 \Rightarrow n$  is even.

The contrapositive is not very helpful here.  
It's easier to express "=" than " $\neq$ "

Ex: Find an integer that cannot be expressed as a difference of two squares (what can you say about this integer before even finding it)