Generalizing even/odd results

Prove: (1) • n even $\Longrightarrow$ n^k even , $k \in \mathbb{N}$

(2) • n odd $\Longrightarrow$ n^k odd , $k \in \mathbb{N} \cup \{0\}$

(1) : n is even $\Rightarrow$ $n = 2m$, $m \in \mathbb{Z}$

$$\Rightarrow n^k = (2m)^k = 2^k \cdot m^k = 2\underbrace{\left(2^{k-1} \cdot m^k\right)}_{\in \mathbb{Z}\,?}$$

Yes

$$= 2 \cdot m' \, , \; m' \in \mathbb{Z}$$

$\Rightarrow n^k$ is even.

(2) : n is odd $\Rightarrow$ $n = (2m+1)$, $m \in \mathbb{Z}$

$$\Rightarrow n^k = (2m+1)^k$$

$$= \underbrace{\binom{k}{0}(2m)^k + \binom{k}{1}(2m)^{k-1} + \cdots \binom{k}{k-1}(2m)^1}_{} + \binom{k}{k}(2m)^0$$

$$2\Bigg[\underbrace{\binom{k}{0}2^{k-1}m^k + \cdots\cdots + \binom{k}{k-1}2^0 m}_{\in \mathbb{Z}}\Bigg] + 1$$

empty sum when $k=0$

$$\Rightarrow n^k \text{ is odd}$$

Conclusion : n is even $\Longleftrightarrow$ n^k is even , $k \in \mathbb{N}$

n is odd $\Longleftrightarrow$ n^k is odd , $k \in \mathbb{N}$

In other words, $n, n^2, n^3, \ldots$ have the same parity !

How do we prove <u>iff</u>

$$P \Leftrightarrow Q$$

1) Prove $P \Rightarrow Q$ is true
2) Prove $Q \Rightarrow P$ is true

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1) Prove $P \Rightarrow Q$ is true
2) Prove $\neg P \Rightarrow \neg Q$ is true

Contrapositive

Prove: There is no smallest positive rational number.

By contradiction:

Let $x > 0$ be the smallest positive rational number

let $y = \frac{x}{2}$ ($y > 0$, $y \in \mathbb{Q}$ because if $x = \frac{a}{b}$ then $y = \frac{a}{2b}$)

contradiction since $y < x$.

Prove : n odd $\Rightarrow$ $n = a^2 - b^2$ where $a, b \in \mathbb{Z}$

n is odd $\Rightarrow$ $n = 2k + 1$, $k \in \mathbb{Z}$

$$= \underbrace{(k+1)^2}_{k^2+2k+1} - k^2$$

Contrapositive: $n \neq a^2 - b^2 \Rightarrow n$ is even.

The contrapositive is not very helpful here.

It's easier to express "$=$" than "$\neq$"

Ex: Find an integer that cannot be expressed as a difference of two squares (what can you say about this integer before even finding it)