Infinity & Beyond

S is Countable set :

- S is Finite, or
- There exists a function $f : S \to \mathbb{N}$ (or $f: \mathbb{N} \to S$) that is a bijection.
  (S and $\mathbb{N}$ have the same cardinality)

Example: $f : \mathbb{N} \to \{2, 4, 6, 8, \ldots\}$

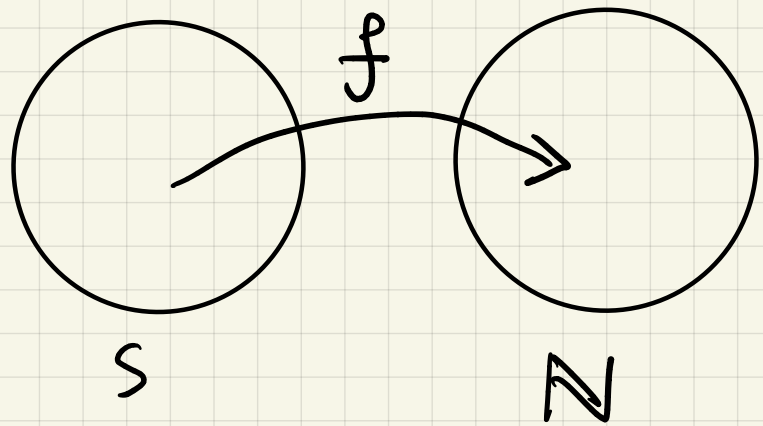
$$f(x) = 2x$$

one-to-one : $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$

onto : $y \in \{2, 4, 6, 8, \ldots\}$, let $x = \frac{y}{2} \in \mathbb{N}$, $f(x) = y$

Alternative definition

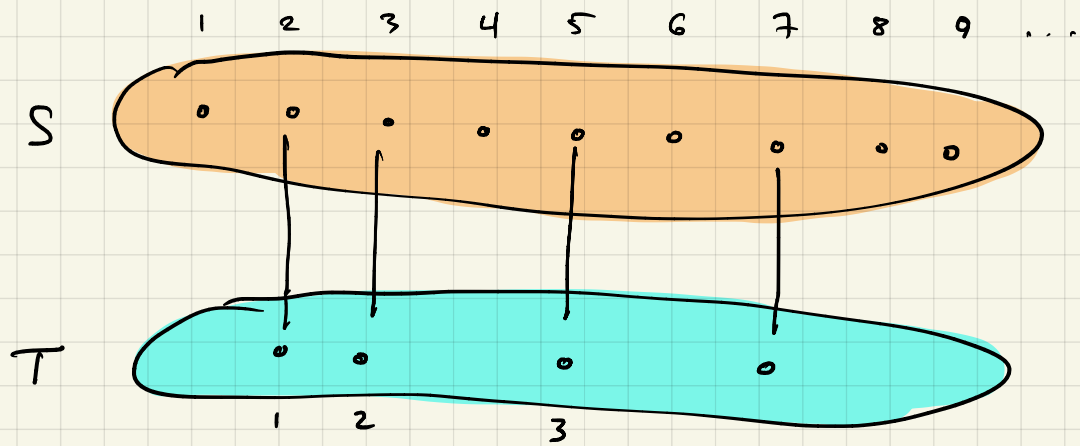
S is countable iff there exist a function $f : S \to \mathbb{N}$ that is one-to-one



Informal interpretation: f "orders" the elements of S.

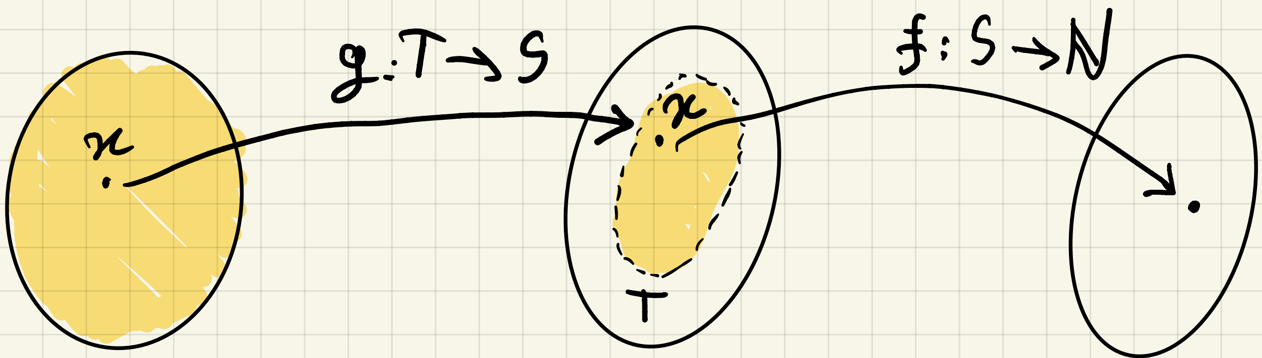
Each element must have a finite rank

If S is countable and $T \subset S$, then T is countable



- Preserve the relative order of elements in T.
- each element in T has a rank that is at most its rank is S.
- each element in T has a finite rank

Alternative view:



T $\quad g(x) = x \quad$ S $\quad f: S \to N \quad$ N

one-to-one

$f \circ g : T \to N$

$f \circ g(x) = f(g(x))$

is one-to-one

Show $f \circ g$ is one-to-one:

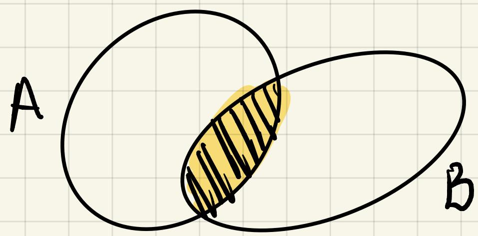
$$f \circ g(x_1) = f \circ g(x_2)$$

$$f(g(x_1)) = f(g(x_2))$$

$$g(x_1) = g(x_2) \qquad \text{(because } f \text{ is one-to-one)}$$

$$x_1 = x_2$$

- If A and B are countable, then $A \cap B$ is countable



$(A \cap B) \subset A$

since A is countable, then

$A \cap B$ is also countable.

- If A and B are countable, then $A \cup B$ is countable

$$A = \{ a_1, a_2, a_3, \cdots \}$$

$$B = \{ b_1, b_2, b_3, \cdots \}$$

rank ?

Consider $A \cup B = \{ a_1, a_2, a_3, \cdots\cdots, b_1, b_2, b_3, \cdots \}$ ✗

$$A \cup B = \{ a_1, b_1, a_2, b_2, a_3, b_3, \cdots\cdots \}$$

$$\left.\begin{array}{l}\text{If } a_i = b_j \text{, drop } b_j \\ \text{from ranking}\end{array}\right\} \Longrightarrow \begin{array}{l}\text{rank of } b_i \leqslant 2i \\ \text{rank of } a_i \leqslant 2i-1\end{array}$$

Is $\mathbb{Z}$ countable?

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

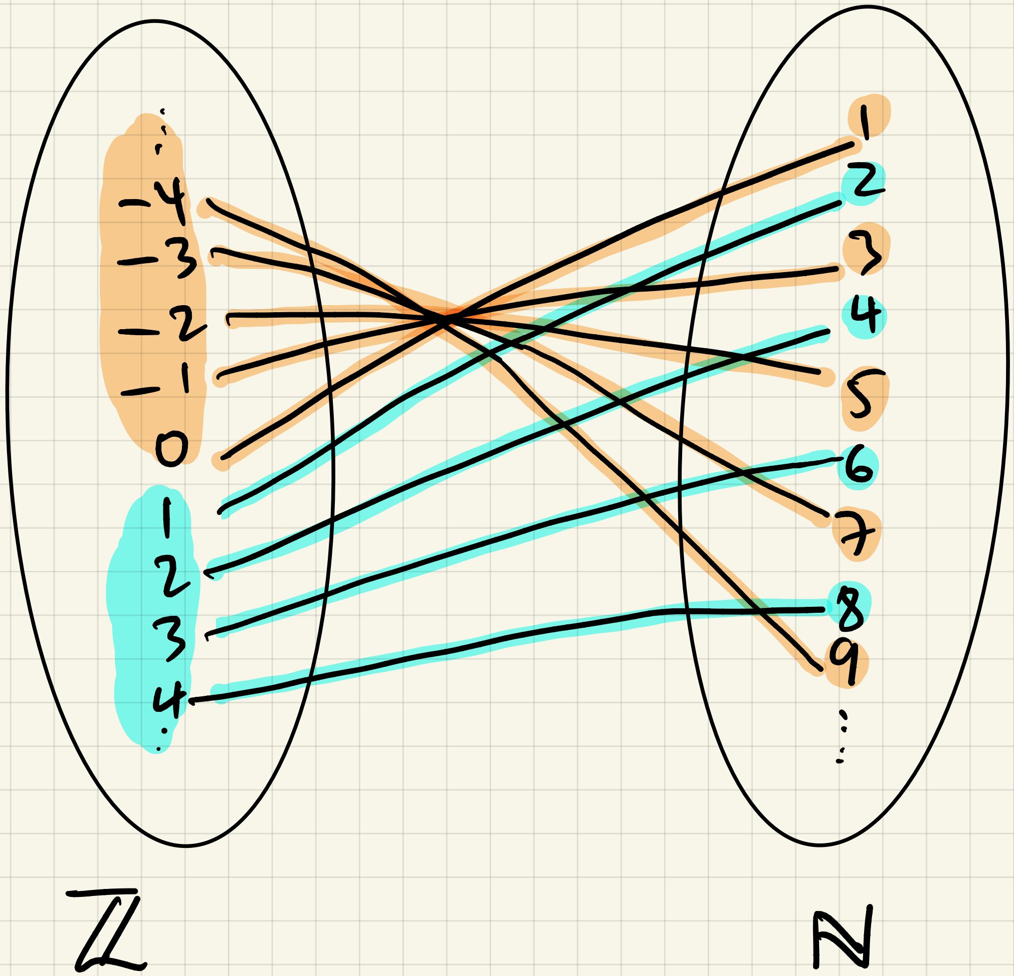
$$\mathbb{Z} = \underbrace{\{0\} \cup \mathbb{N}}_{\text{countable}} \cup \underbrace{\{-1, -2, -3, \dots\}}$$

$f: \{-1, -2, -3, \dots\} \to \mathbb{N}$

$f(x) = -x$

is a bijection

countable



Find a bijection

$f: \mathbb{Z} \to \mathbb{N}$

$$f(x) = \begin{cases} 2x & x > 0 \\ -2x + 1 & x \leq 0 \end{cases}$$

(Bijection)

One-to-one:

$$f(x_1)=f(x_2) \Rightarrow 2x_1 = -2x_2+1 \quad (\text{even} = \text{odd})$$

$$2x_1 = 2x_2 \Rightarrow x_1 = x_2$$

$$-2x_1+1 = 2x_2$$

$$-2x_1+1 = -2x_2+1 \Rightarrow x_1 = x_2$$

Onto: Given $y \in \mathbb{N}$,

- if y is even, let $x = \frac{y}{2} \in \mathbb{Z}_{>0}$ $(y \geq 2)$. $f(x) = 2\frac{y}{2} = y$.
- if y is odd, let $x = \frac{1-y}{2} \in \mathbb{Z}_{\leq 0}$ $(y \geq 1)$.

$$f(x) = -2\left(\frac{1-y}{2}\right)+1 = y.$$