Is $\mathbb{Z}$ countable?

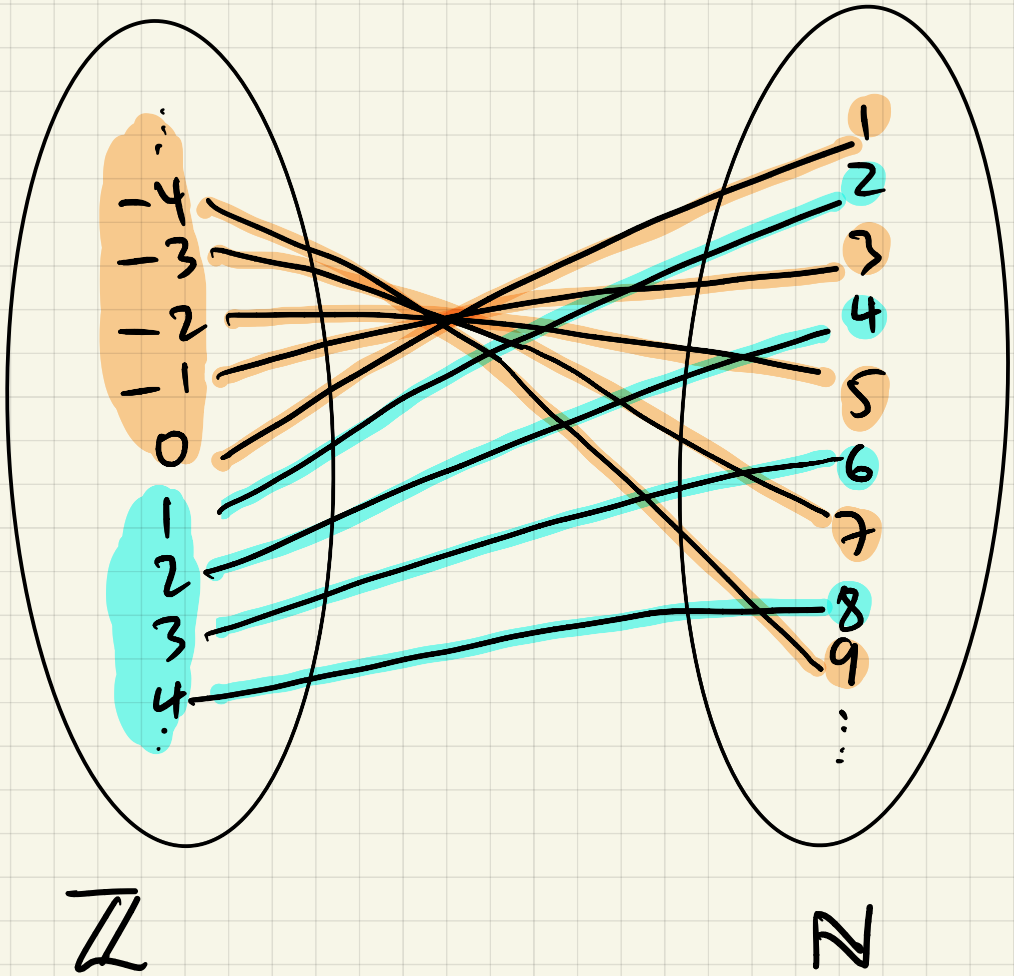
$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

$$\mathbb{Z} = \underbrace{\underbrace{\{0\} \cup \mathbb{N}}_{\text{countable}} \cup \underbrace{\{-1, -2, -3, \ldots\}}}_{\text{countable}}$$

$f : \{-1, -2, -3, \ldots\} \to \mathbb{N}$

$f(x) = -x$

is a bijection



Find a bijection

$$f : \mathbb{Z} \to \mathbb{N}$$

$$f(x) = \begin{cases} 2x & x > 0 \\ -2x + 1 & x \leq 0 \end{cases}$$

(Bijection)

One-to-one:

$$f(x_1) = f(x_2) \Rightarrow 2x_1 = -2x_2 + 1 \quad \times \quad (\text{even} = \text{odd})$$

$$\text{or} \quad 2x_1 = 2x_2 \Rightarrow x_1 = x_2$$

$$\text{or} \quad -2x_1 + 1 = 2x_2 \quad \times$$

$$\text{or} \quad -2x_1 + 1 = -2x_2 + 1 \Rightarrow x_1 = x_2$$

Onto: Given $y \in \mathbb{N}$,

- if y is even, let $x = \frac{y}{2} \in \mathbb{Z}_{>0}$ $(y \geq 2)$. $f(x) = 2\frac{y}{2} = y$.
- if y is odd, let $x = \frac{1-y}{2} \in \mathbb{Z}_{\leq 0}$ $(y \geq 1)$.

$$f(x) = -2\left(\frac{1-y}{2}\right) + 1 = y.$$

Is $\mathbb{Q}$ countable?

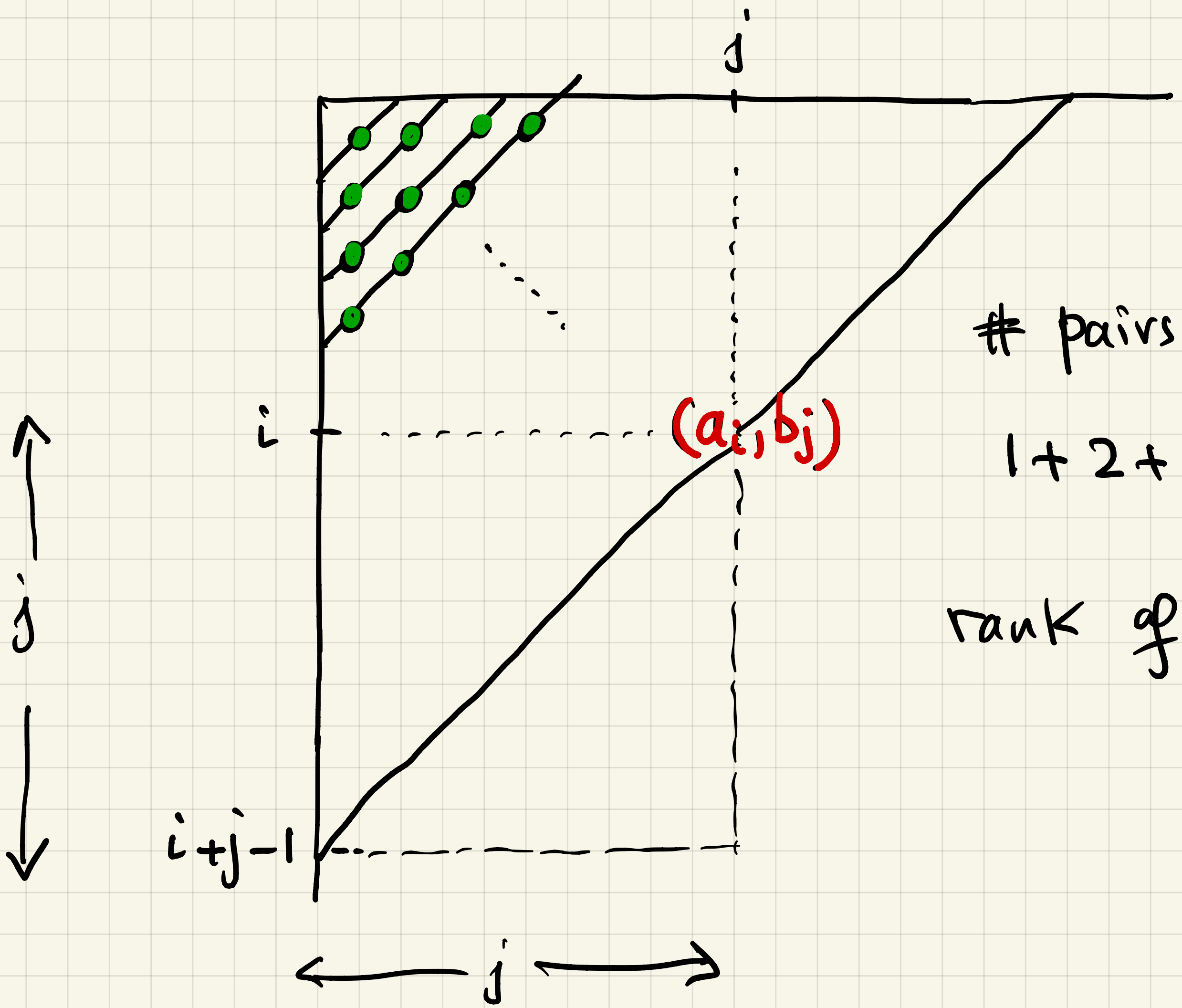
If A and B are countable, then $A \times B$ is countable

$A = \{a_1, a_2, a_3, \dots\}$ $B = \{b_1, b_2, b_3, \dots\}$

	b_1	b_2	b_3	b_4	$\dots$
a_1	(a_1, b_1)	(a_1, b_2)	$\dots$		
a_2	(a_2, b_1)	$\ddots$			
a_3	$\vdots$				
a_4					
$\vdots$					

we can "think" of $\mathbb{Q}$ as a "subset" of $\mathbb{Z} \times \mathbb{N}$

$\frac{a}{b}$ "as" (a, b)



pairs in the triangle

$$1+2+3+\cdots+(i+j-1)$$

rank of $(a_i, b_j) \leq 1+2+\cdots+(i+j-1)$

$$= \frac{(i+j-1)(i+j)}{2}$$

$$(i+j-1) - i + 1 = j$$

Summary:

To show that an infinite set S is countable

Formal way: Show there exist $f: S \to \mathbb{N}$ that is a bijection

Informal way: order the elements of S such that each will have a finite rank

Facts:
- S is countable, any subset of S is countable.
- A and B are countable, then $A \cup B$, $A \cap B$, $A \times B$ are countable.

$\mathbb{R}$ is uncountable (Yay!)

There is NO bijection f

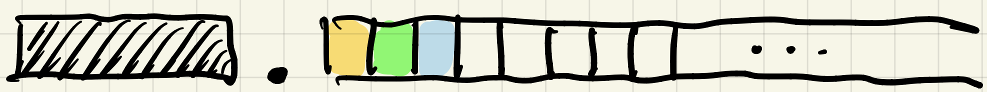
$$f : \mathbb{N} \to \mathbb{R}$$

Proof by contradiction: Cantor's diagonal proof

Let $f : \mathbb{N} \to \mathbb{R}$. We will construct $x \in \mathbb{R}$ such that no $i \in \mathbb{N}$ satisfies $f(i) = x$.

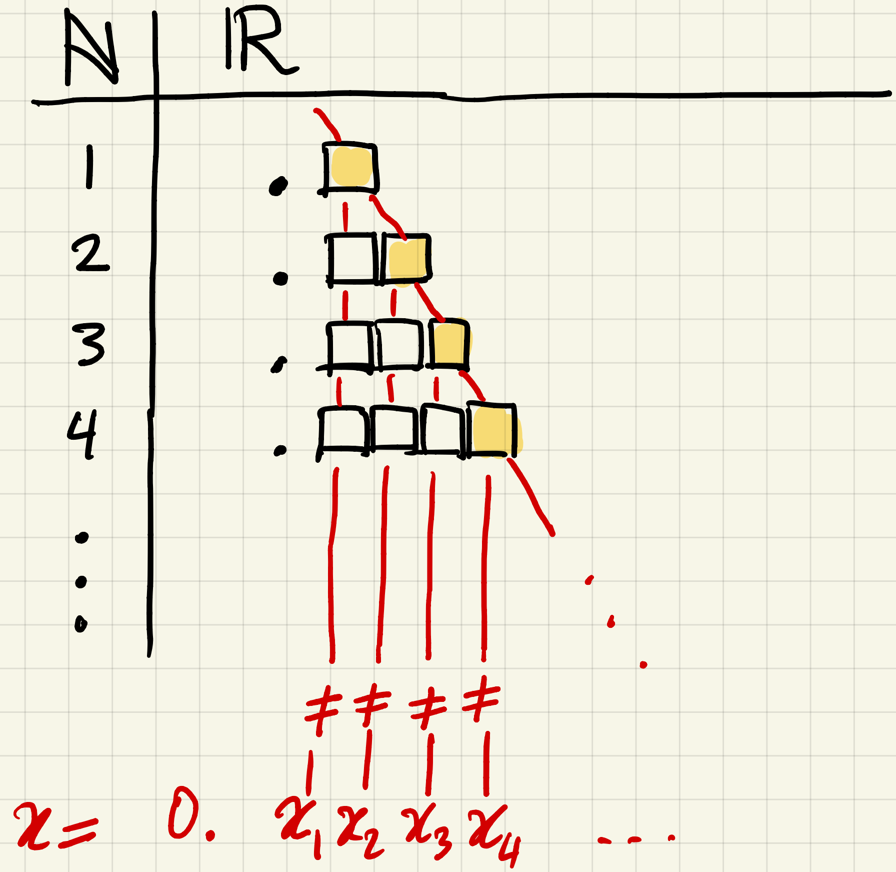
So if f is a bijection, we have a contradiction (it's not onto).

First for every $r \in \mathbb{R}$, we will refer to the i^{th} digit of r as the i^{th} digit following the decimal point in r's representation.

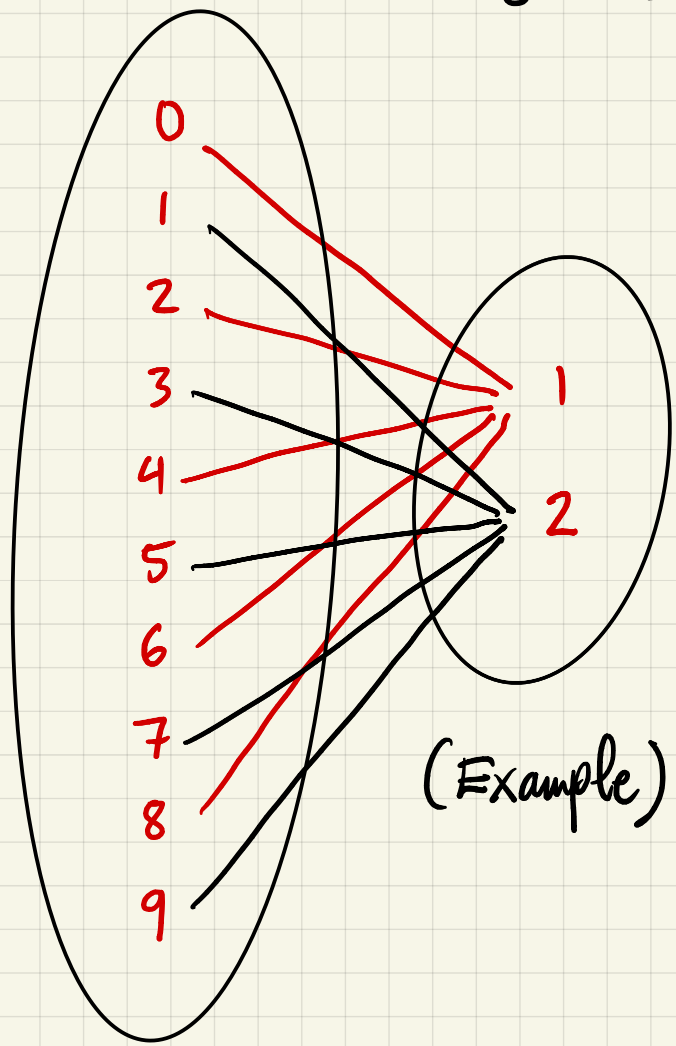
r 

We will make $x = 0.x_1 x_2 x_3 \ldots$

where digit $x_i \neq \underbrace{i^{th} \text{ digit of } \overbrace{f(i)}^{\in \mathbb{R}}}_{\text{well defined concept}}$



how to change digits?



(Example)

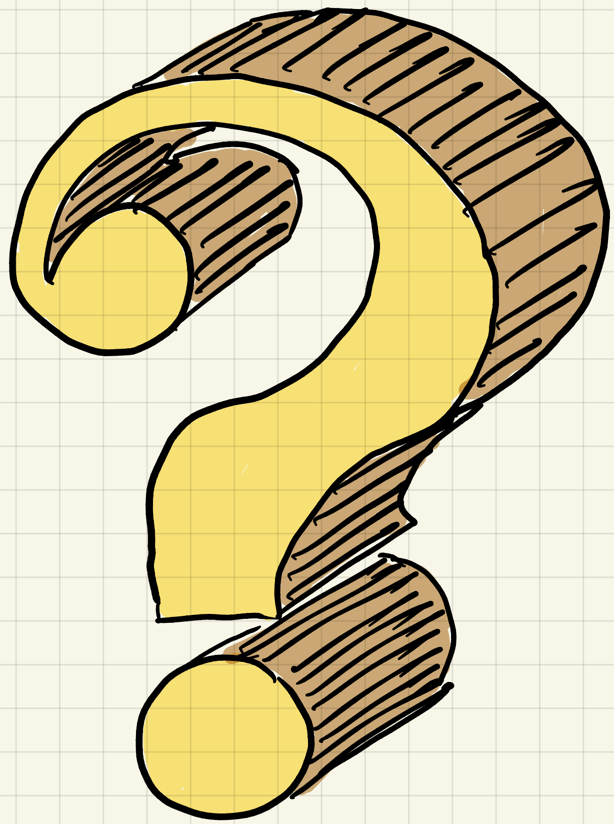
There is no $i \in \mathbb{N}$ such that $f(i) = x$ because x is different from $f(i)$ in the ith digit.

"Example" diagonalization

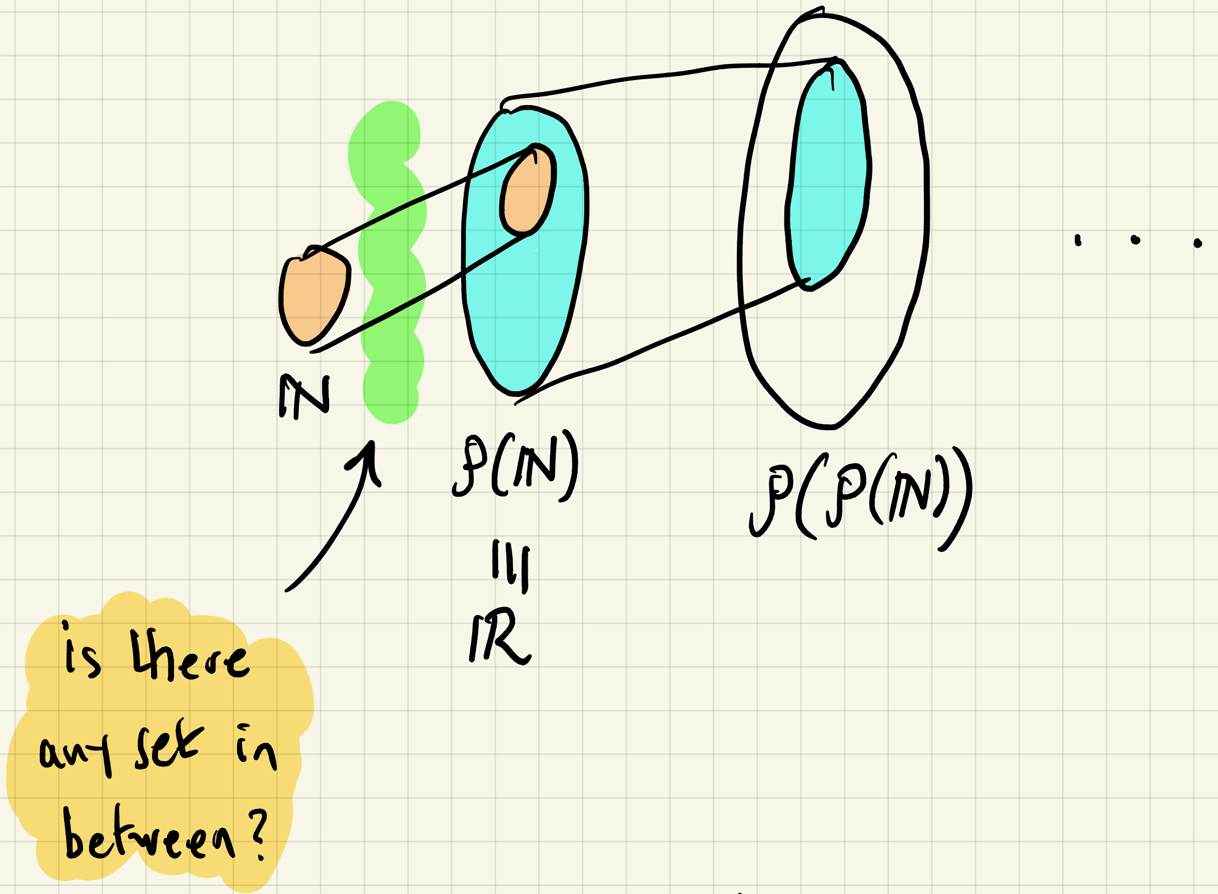
$\mathbb{N}$	$\mathbb{R}$
1	0.5000000...
2	0.1415...
3	0.7130000...
4	0.860100...
5	0.6666666...

$x = 0.21221 \ldots$

Question : If we attempt to prove that $\mathbb{Q}$ is uncountable (which is false), where does Cantor's diagonalization proof break



Given any set S, using a similar diagonal proof we can show there is no bijection from S to $\mathcal{P}(S)$



"the continuum hypothesis" is independent of the axioms of set theory (can't prove it or disprove it!)