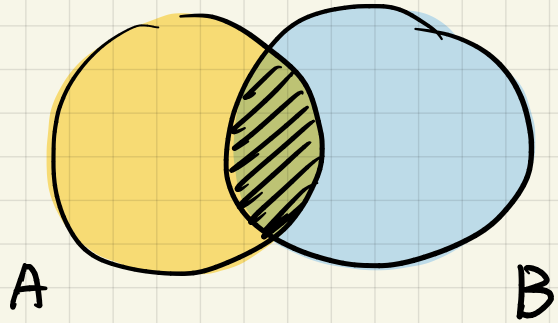
Inclusion - Exclusion

Basic setting:

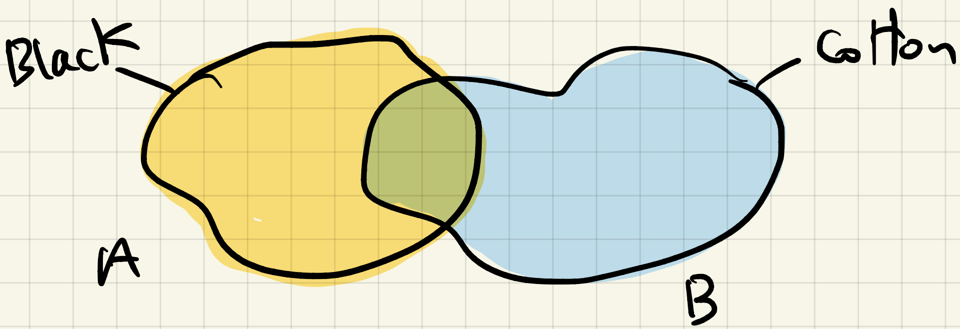


$$|A \cup B| = |A| + |B| - |A \cap B|$$

(why ?)

Example: 50 socks — 35 black, 30 cotton

How many are black and cotton ?

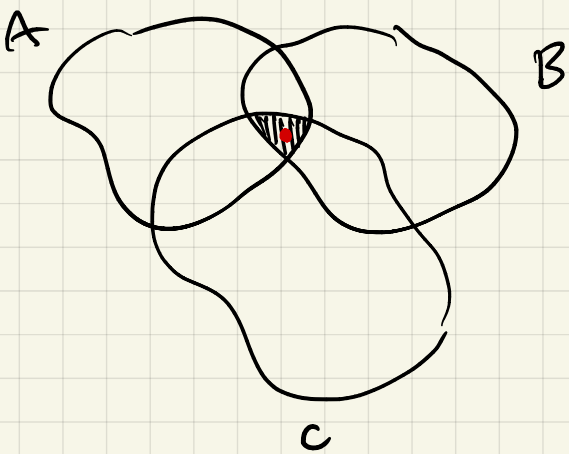


$$|A \cup B| = |A| + |B| - \overbrace{|A \cap B|}^{?}$$

$$50 = 35 + 30 - |A \cap B|$$

$$|A \cap B| = 15$$

What about multiple sets



$$|A \cup B \cup C| = |A| + |B| + |C|$$
$$- |A \cap B| - |A \cap C| - |B \cap C|$$
$$+ |A \cap B \cap C|$$

Four sets:

\# terms

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| \quad \text{(include)} \qquad \binom{4}{1} = 4$$
$$- |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \quad \text{(exclude)} \qquad \binom{4}{2} = 6$$
$$+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \quad \text{(include)} \qquad \binom{4}{3} = 4$$
$$- |A \cap B \cap C \cap D| \quad \text{(exclude)} \qquad \binom{4}{4} = 1$$

Why it works ?

Consider an element that is in n sets.

- It belongs to $\binom{n}{1}$ sets
- It belongs to $\binom{n}{2}$ pairs of sets.
- It belongs to $\binom{n}{3}$ triplets of sets

$\vdots$

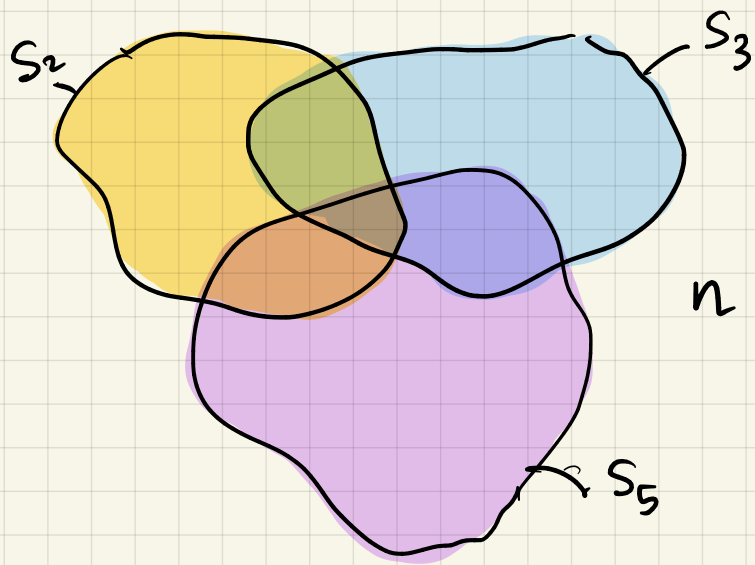
- It belongs to $\binom{n}{n}$ n-tuple of sets

$$\#\text{times element is added} = \binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \binom{n}{4} + \dots + (-1)^{n-1}\binom{n}{n}$$

$$= \binom{n}{0} - \left[\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + (-1)^n\binom{n}{n}\right]$$

$$= 1 - 0^n = \begin{cases} n=0: & 1-0^0 = 1-1=0 \\ n>0: & 1-0^n = 1-0 = \boxed{1} \end{cases}$$

Example: How many positive integers ≤ 1000 are divisible by 2 or 3 or 5?



if p, q are prime:

n divisible by both $\Longleftrightarrow$ n divisible by pq

$$|S_2 \cup S_3 \cup S_5| = |S_2| + |S_3| + |S_5| - |S_2 \cap S_3| - |S_2 \cap S_5| - |S_3 \cap S_5| + |S_2 \cap S_3 \cap S_5|$$

$$= \left\lfloor \frac{1000}{2} \right\rfloor + \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{5} \right\rfloor - \left\lfloor \frac{1000}{6} \right\rfloor - \left\lfloor \frac{1000}{10} \right\rfloor - \left\lfloor \frac{1000}{15} \right\rfloor + \left\lfloor \frac{1000}{30} \right\rfloor = ?$$

How many passwords of length n are there

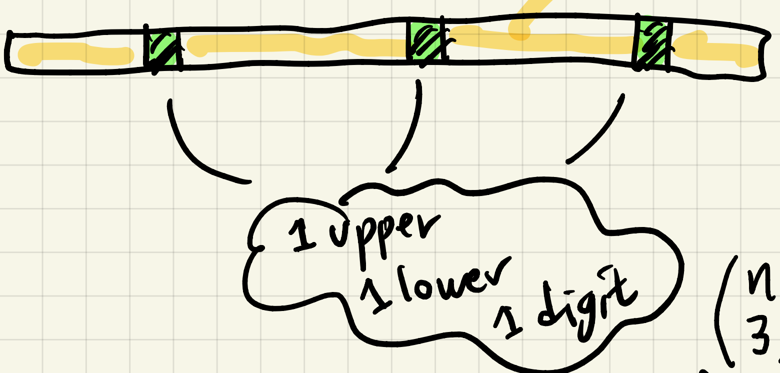
$$\Sigma = \{A, \dots, Z, a, \dots, z, 0, \dots, 9\}$$

Answer: 62^n (select n from 62 with order & rep.)

Good password has at least one upper case **and** one lower case **and** one digit

62^{n-3}

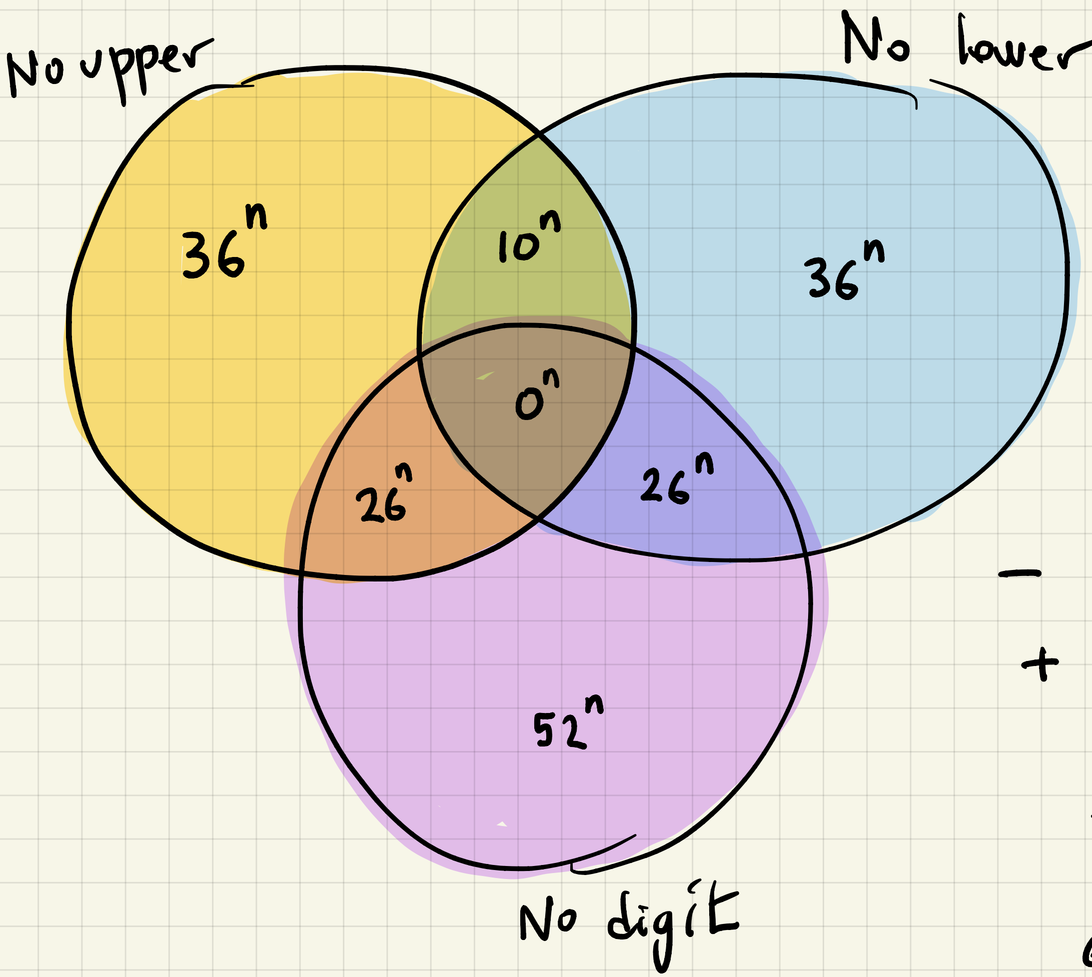
product rule overcounts



$$\underbrace{\binom{n}{3} \times 3!} \times 26 \times 26 \times 10$$

1) Choose 3 pos. with order
2) Assign them upper, lower, digit
3) Choose rest in 62^{n-3} ways

Bad password



bad words:

$$36^n + 36^n + 52^n - 26^n - 26^n - 10^n + 0^n$$

good words:

$$62^n - (\sim)$$

Note: When $n=3$, verify that you get $\binom{n}{3} \times 3! \times 26 \times 26 \times 10$

Lazy Professor

How many permutations of $(1, 2, 3, \ldots, n)$ are there if every number i does not occur in position i

"Derangements"

This is an "AND" logic.

Initial attempt: (Fails)

		# ways
1. Choose a position for 1	----	$(n-1)$
2. Choose a position for 2	---	?

$(n-2)$?

$(n-1)$?

Can't tell!

Depends on choice for 1