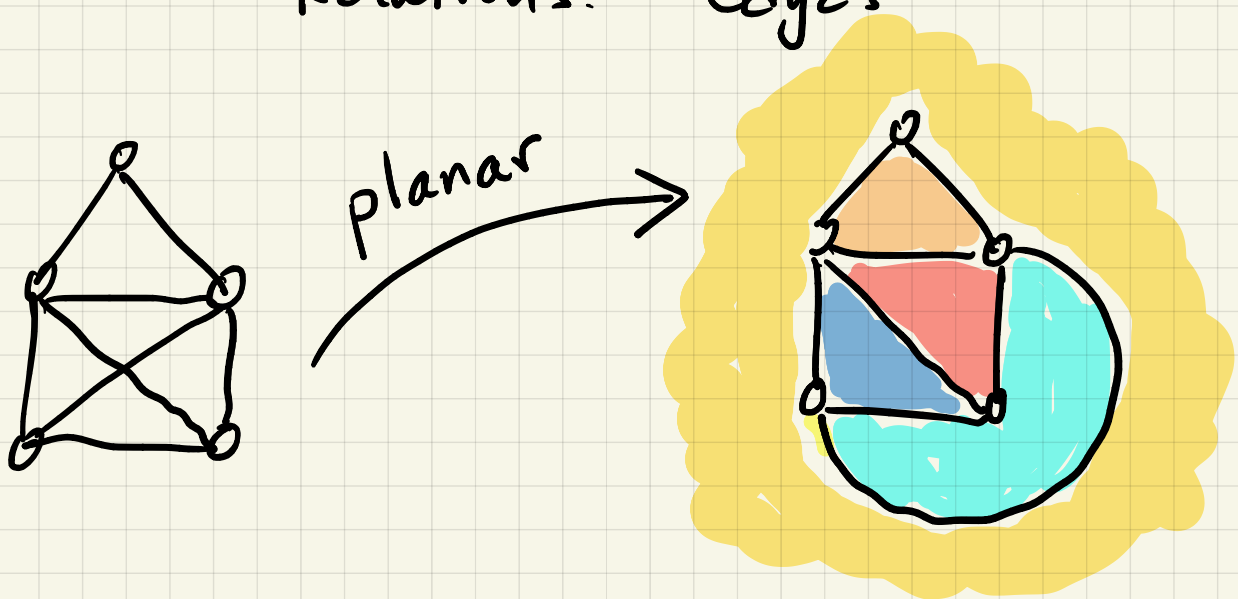
Graphs : Pairwise Relation

Entities: vertices

Relations: Edges



$v =$ 5 (vertices)

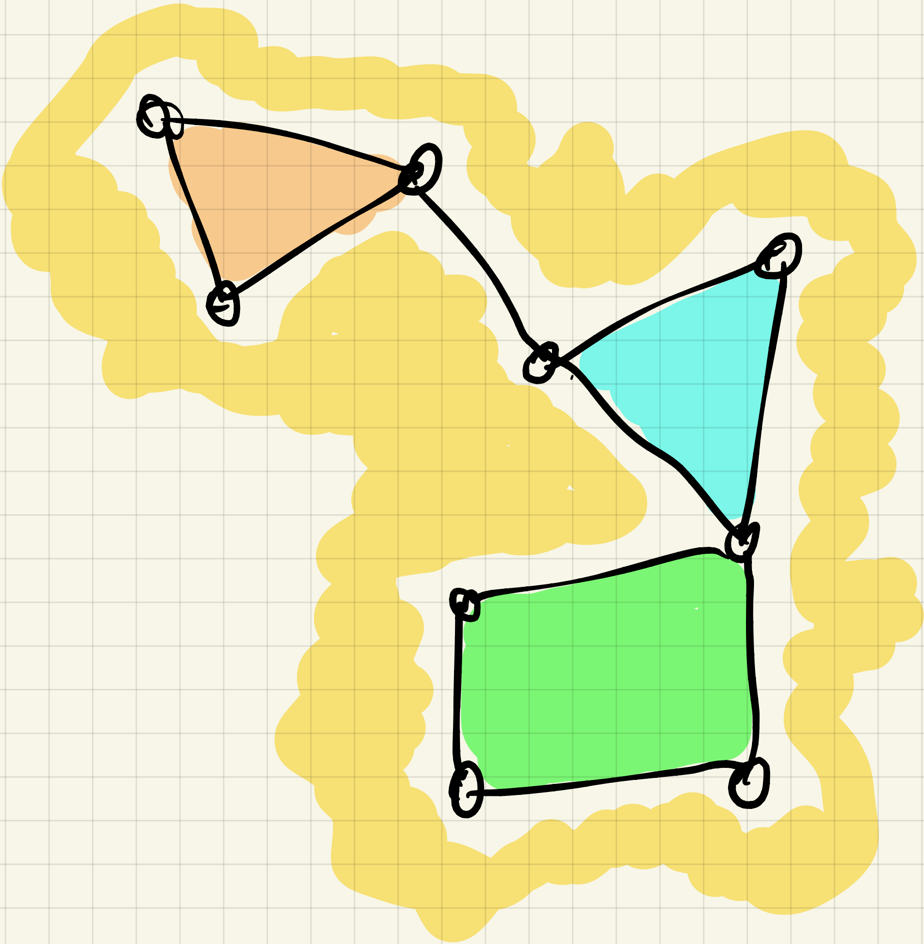
$e =$ 8 (edges)

$f =$ 5 (faces)

Counting helps establish patterns / facts

face: area we can move without crossing any edge (defined only for planar graphs)

Euler : $v - e + f = 2$ (planar graphs)



$v = 9$

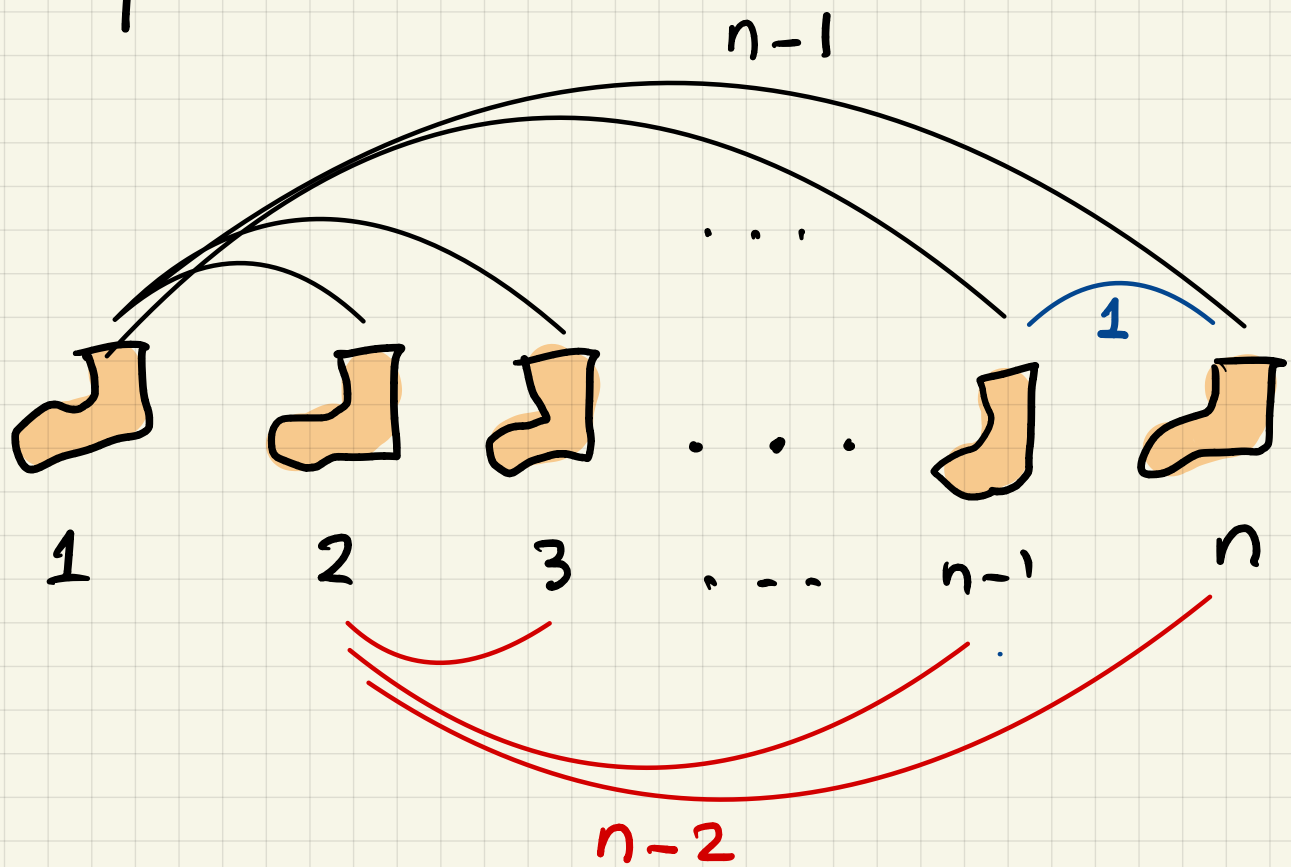
$e = 11$

$f = 4$

$v - e + f = 9 - 11 + 4 = 2$

Another Example: n socks

In how many ways can we make a pair?



Total $= 1 + 2 + 3 + \cdots + (n-1)$

Recall:

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

Pattern: $1+2+3+\cdots+\square=\frac{\square(\square+1)}{2}$

Then: $1+2+3+\cdots+(n-1)=\frac{(n-1)n}{2}$

Example: 6 socks, $n=6$.

$$\#\text{pairs}=\frac{(6-1)6}{2}=\frac{5\times 6}{2}=15$$

"n choose 2"

$$\#\text{pairs on } n \text{ things}=\frac{n(n-1)}{2}=\binom{n}{2}$$

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

Let's generalize even further:

- Start with a.
- End in b
- Jump by s

$$S = a + (a+s) + (a+2s) + \dots + b$$

$$S = \text{avg}(a, b) \times \#\,\text{terms} = \left(\frac{a+b}{2}\right)\overbrace{\left(\underbrace{\frac{b-a}{s}}_{\#\,\text{jumps}} + 1\right)}^{\#\,\text{terms}}$$

Lazy Professor: Does not want to grade, permutes the tests

Example: # students $= n = 3$

A	B	C	
A	B	C	✗
A	C	B	✗
B	A	C	✗
B	C	A	✓
C	A	B	✓
C	B	A	✗

permutations

A permutation defines an "order" on the objects

Counting also helps understand the complexity of objects we are dealing with. (see below)

permutations on n objects

$n! = 1 \times 2 \times 3 \times \dots \times n$

"n factorial"

"factorial of n"

Examples:

$n=3$: $3! = 1 \times 2 \times 3 = 6$

$n=4$: $4! = 1 \times 2 \times 3 \times 4 = 24$

$n=5$: $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$

$\vdots$

$n=10$: $10! = 1 \times 2 \times 3 \times \dots \times 10 = 3628800$

$\vdots$

$n=100$: $100!$ is a 158 digit number

grows very fast

Summation & Product notations

Consider this: add the first 10 terms of:

$$1 + 2 + 4 + \overset{?}{\cdots}$$

what do you mean? (Ambiguous)

$$\underset{+1}{\curvearrowright}\ \underset{+2}{\curvearrowright}\ \underset{+3}{\curvearrowright}\ 7$$

What I wanted: $1 + 2 + 4 + 8 + 16 + \dots$ (powers of 2)

Sum notation: $\displaystyle\sum_{i=0}^{9} 2^i = 2^0 + 2^1 + 2^2 + \dots + 2^9$

precise

upper bound

$$\sum_{i=a}^{b} f(i) \qquad\qquad \prod_{i=a}^{b} f(i)$$

lower bound

Evaluate $f(i)$ for $i=a, a+1, \dots, b$, then

Add them up

Multiply them

Naming things and using notation makes it easy to communicate ideas and eliminates ambiguity.

e.g. Archimedes named π, before that people used different values for it!

Examples:

$$\sum_{i=1}^{n-1} i = 1+2+3+\cdots+(n-1) = \binom{n}{2} = \#\text{ pairs}$$

$$\sum_{i=1}^{n} (2i-1) = \underbrace{1+3+5+\cdots+(2n-1)}_{\text{what's this in English?}}$$

$$\prod_{i=1}^{n} i = 1\times 2\times 3\times \cdots \times n = n! = \#\text{ permutations}$$

What if $n=0$?

$$\sum_{i=1}^{0} f(i) = \text{Empty sum} = 0$$

$$\prod_{i=1}^{0} f(i) = \text{Empty product} = 1$$

Rewrite sum & product Notation as programs (loops)

$$\sum_{i=a}^{b} f(i)$$

$$\prod_{i=a}^{b} f(i)$$

```
s = ?
i = a
while i ≤ b :
    s = s + f(i)
    i = i + 1
return s
```

```
p = ?
i = a
while i ≤ b :
    p = p * f(i)
    i = i + 1
return p
```

- How should s, p be intitialized ?
- What happens when $b < a$ in each case ?