Lazy Professor

How many permutations of $(1, 2, 3, \ldots, n)$ are there if every number i does not occur in position i

"Derangements"

This is an "AND" logic.

Initial attempt: (Fails)

		# ways
1.	Choose a position for 1 ----	$(n-1)$
2.	Choose a position for 2 ---	?

$(n-2)$?

$(n-1)$?

Can't tell!

Depends on choice for 1

Bad permutation:

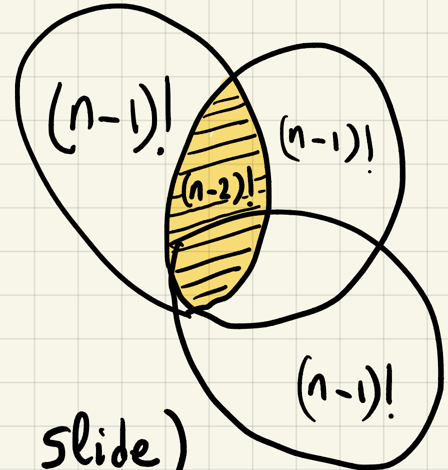
1 is in the first position, OR

2 is in the second position, OR

3 is " " third position, OR

⋮

n is in the n^{th} position.



(see next slide)

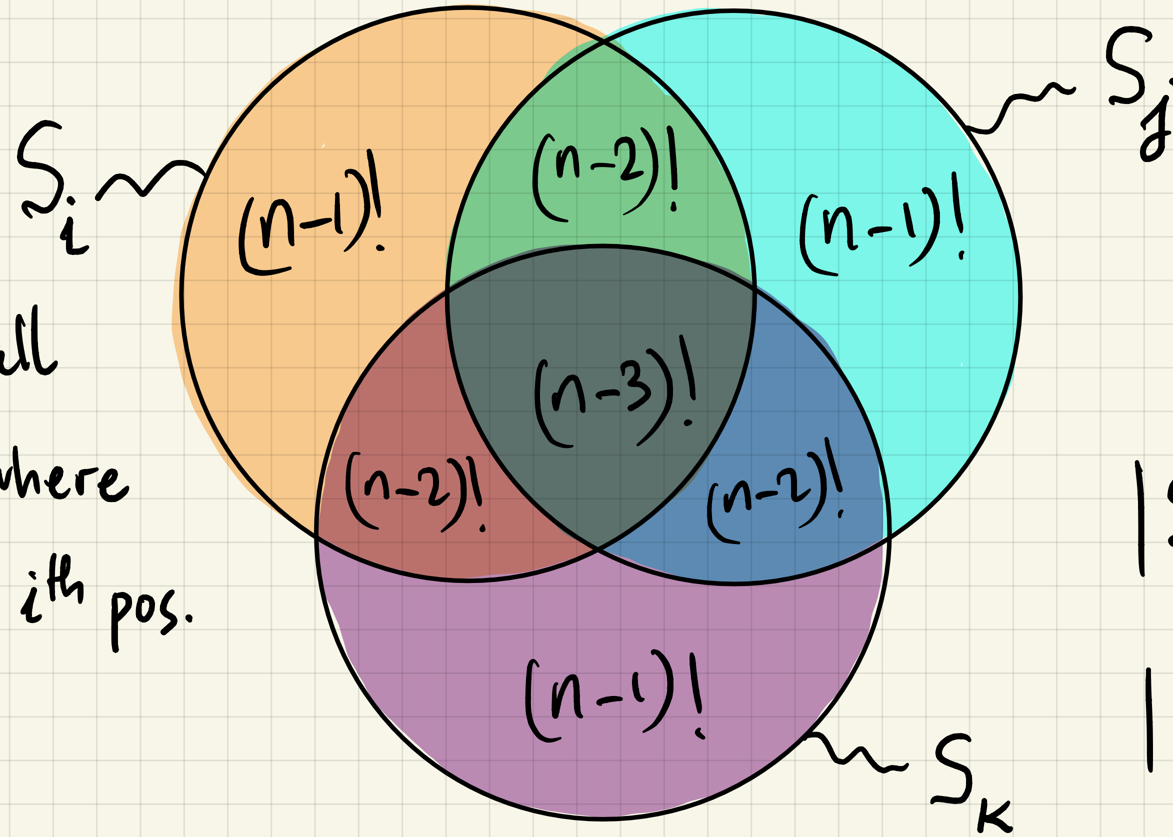
$$n\cdot(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! \; ----- \; \binom{n}{n}(n-n)!$$

$$\#\text{ bad}: \quad \frac{n!}{1!} - \frac{n!}{2!} + \frac{n!}{3!} - \cdots + (-1)^{n-1}\frac{n!}{n!}$$

$$\#\text{ good}: \quad \frac{n!}{0!} - \left[\frac{n!}{1!} - \frac{n!}{2!} + \frac{n!}{3!} - \cdots \cdots + (-1)^{n-1}\frac{n!}{n!}\right]$$

$$= n!\left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n\frac{1}{n!}\right] \quad \left(e^x = \sum_{i=0}^{\infty}\frac{x^i}{i!}\right)$$

$$n \text{ large} \Rightarrow \approx n!\, e^{-1} = \frac{n!}{e}$$



set of all perm. where i is in i^{th} pos.

$|S_i| = (n-1)!$

$|S_i \cap S_j| = (n-2)!$

$|S_i \cap S_j \cap S_k| = (n-3)!$

There are $\binom{n}{k}$ intersection of k sets, each has $(n-k)!$ permutations.

derangement:

$$\#\text{derangements} = \underbrace{!n}_{\text{Notation}} = \left\lfloor \frac{n!+1}{e} \right\rfloor \quad n \geq 1$$

$!0 = 1$

$!1 = 0$

$!2 = 1$

$!3 = 2$

$!4 = 9$

$!5 = 44$

$\vdots$

$\lfloor x \rfloor =$ the largest integer $\leq x$

Example: $\lfloor 5.3 \rfloor = 5$

Pigenhole principle

- Choose 51 numbers from $\{1, 2, \ldots, 100\}$. Prove that two of them are consecutive.

- Place 10 points inside a 3×3 square. Prove that two of them will be within a distance of $\sqrt{2}$

- Place numbers $1, 2, \ldots, 10$ in 3 bins. Prove that sum in one bin is at least 19.

Freedom in doing something, Guaranteed Consequence

Pigeonhole

<u>Pigeon hole:</u> Given $n+1$ objects and n boxes, if we place all objects in boxes, at least one box will contain at least 2 objects.

[Basic form]

<u>proof:</u> (By contradiction)

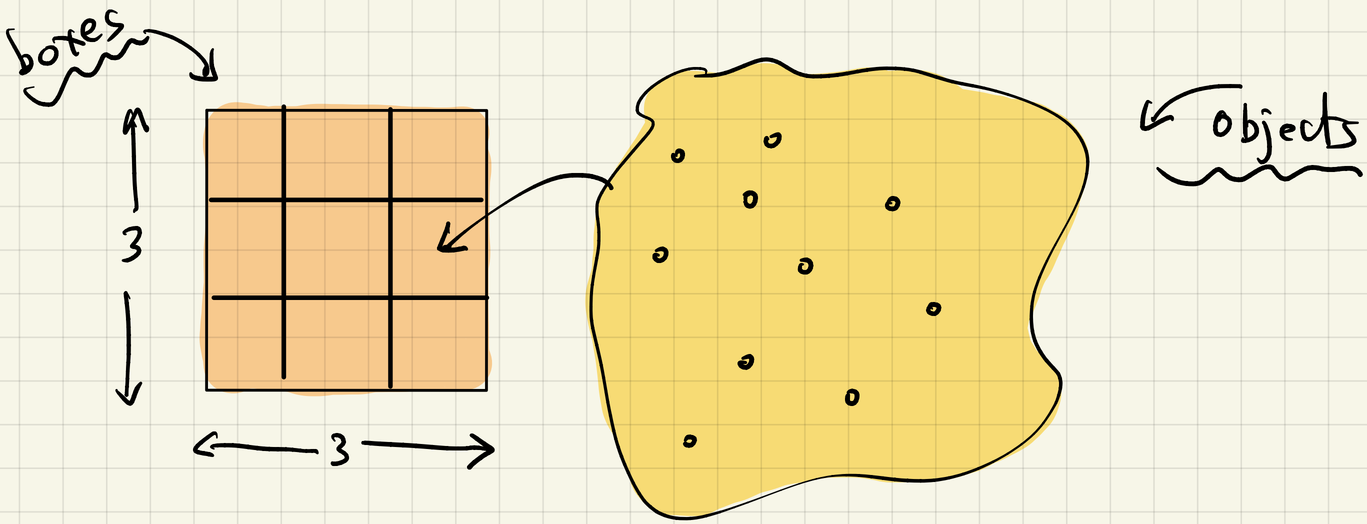
Each boxes has at most one objects

Let x_i be # objects in box i, then $x_i \leq 1$

Total number of objects $= \sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$

$$\leq 1 + 1 + \cdots + 1 = n$$

a contradiction, since we have $n+1$ objects.



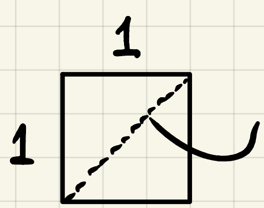
Prove two points will be within a distance $\sqrt{2}$

10 objects
9 boxes

$\Rightarrow$ Pigeonhole: 2 objects will be in the same box

$\Downarrow$

2 points are in the same 1x1 square



longest distance inside square

By Pythagoras, it's $\sqrt{2}$

Proof by Pigeonhole

1) Think Pigeonhole

2) Set in up
   - what are the objects
   - what/where are the boxes

3) Interpret the Pigeonhole result in the given context