Proof by Pigeonhole

1) Think Pigeonhole

2) Set in up
   - what are the objects
   - what/where are the boxes

3) Interpret the Pigeonhole result in the given context

Choose 51 numbers in $\{1, 2, \ldots, 100\}$

Prove 2 of them will be consecutive

Avoid "mehanistic" arguments

1 2 3 4 5 6 7 ---- 99 100

can't do it: only 50

Pigenhole argument:

$\boxed{1,2}$ $\boxed{3,4}$ $\boxed{5,6}$ ---- $\boxed{99,100}$

To choose a number, place a token in its box

51 tokens, 50 boxes, done! One box must contain at least 2 tokens $\Rightarrow$ we have 2 consecutive numbers.

A bit more general

Pigeonhole: If we have n boxes, and m objects and we place all objects in boxes, then at least one box will have at least

$$\left\lfloor \frac{m-1}{n} \right\rfloor + 1 \text{ objects.}$$

Proof: similar to previous proof (by contradiction)

If $n, m \in \mathbb{N}$, $\left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lceil \frac{m}{n} \right\rceil$

"some box must contain at least the average"

$$x-1 < \lfloor x \rfloor \leqslant x \leqslant \lceil x \rceil < x+1$$

- Assume every box has at most $\left\lfloor \frac{m-1}{n} \right\rfloor$ objects

Total # objects $\leqslant n\left\lfloor \frac{m-1}{n} \right\rfloor \leqslant n \, \frac{m-1}{n} = m-1$,

a contradiction.

- Assme every box has at most $\left\lceil \frac{m}{n} \right\rceil - 1$ objects

Total # objects $\leqslant n\left(\left\lceil \frac{m}{n} \right\rceil - 1\right) < n\left(\frac{m}{n} + 1 - 1\right) = m$,

a contradiction

Which one is better to use? Both:

$$m, n \in \mathbb{N} \Rightarrow \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lceil \frac{m}{n} \right\rceil$$

An alternative way to look at Pigeonhole

Given n numbers $x_1, x_2, \ldots, x_n$ such that

$$x_1 + x_2 + \ldots + x_n = m$$

Then $\exists\, x_i .\ x_i \geq \frac{m}{n}$

("One number must be at least the average")

We know this

x_i : # objects in box i

n : # boxes

m : # objects

But if $x_i \geq \frac{m}{n}$, then $x_i \geq \left\lceil \frac{m}{n} \right\rceil$

Application: Place the numbers $1, 2, \ldots, 10$ in 3 bins.

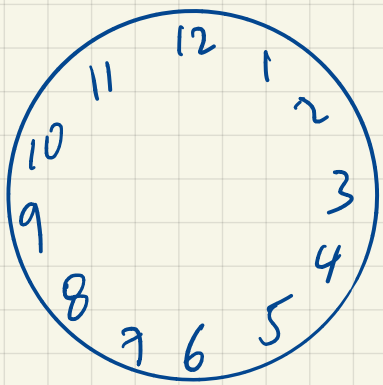
- Prove at least one bin contains at least 4 numbers.

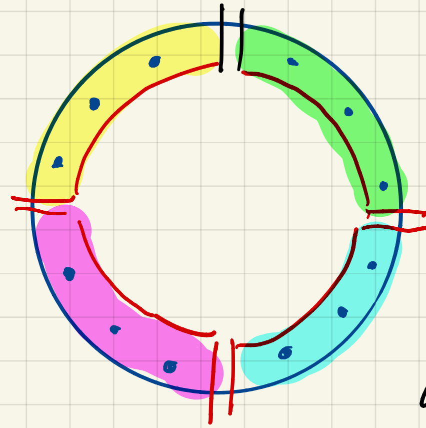
Pigeonhole: $n = 3$, $m = 10$. $\left\lceil \frac{m}{n} \right\rceil = \left\lceil \frac{10}{3} \right\rceil = 4$

- Prove at least one bin contains a sum of at least 19.

Pigeonhole: $n = 3$, $m = 1 + 2 + \cdots + 10 = 55$. $\left\lceil \frac{55}{3} \right\rceil = 19$

Scrambled Clock: Scramble the numbers. Show there are 3 consecutive numbers that add up to at least 20





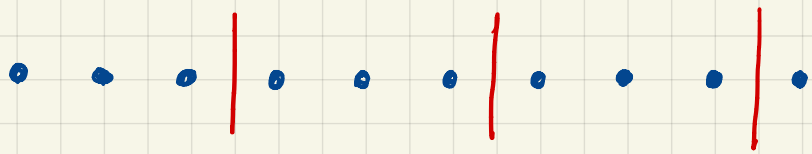
"Divide" the clock into boxes of 3.

One box must contain a sum of at least

$$\left\lceil (1 + 2 + \cdots + 12)/4 \right\rceil = 20.$$

- Sometimes, we have to be "smart" about how to set up the problem.

- Here's an example: The numbers 1 to 10 are written down in some order. Show there are 3 consecutive numbers that add up to at least 15.



One of the 4 boxes must contain a sum of at least

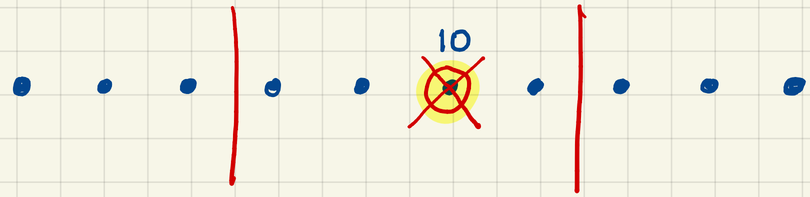
$$\left\lceil \frac{55}{4} \right\rceil = \mathbf{14}.$$

Does not give us what we want!!!

Idea: 4 boxes too many.

Remove largest number, 10.

example:



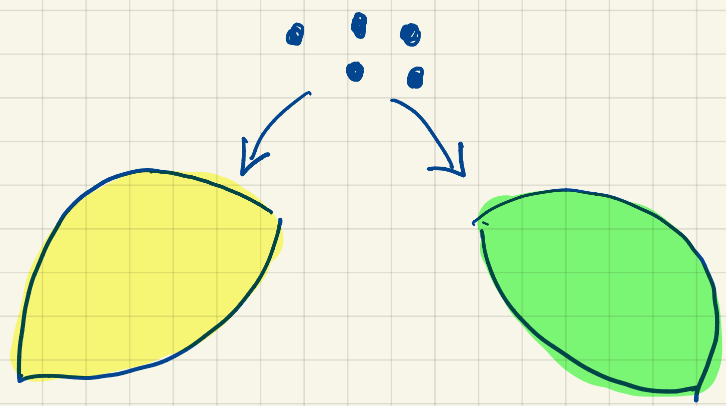
One box must contain a sum of at least $\left\lceil \frac{45}{3} \right\rceil = 15$.

But what if it's the box with a 10?

- Not consecutive any more
- Replace last number in box by 10
- Since 10 is largest, still works

Another example of being "smart" in setting up the pigeonhole.

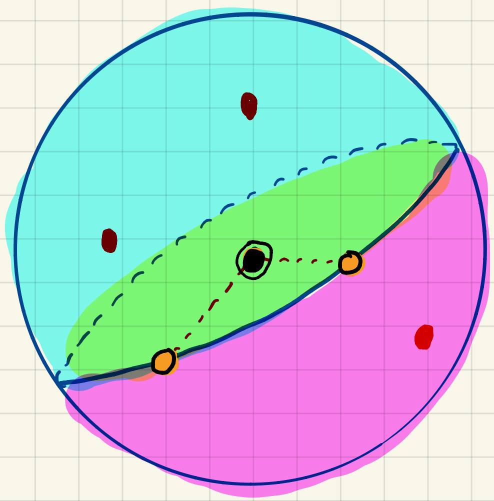
Place 5 points on surface of sphere. Prove that 4 of them lie in the same hemisphere.



By Pigeonhole one hemisphere must contain $\lceil \frac{5}{2} \rceil = 3$ points.

Does not give us what we want!

Idea: Choose the hemispheres.



Choose hemispheres defined by the center and 2 points, as shown.

Pigeonhole: Among the 3 remaining points, one hemisphere will contain at least $\lceil \frac{3}{2} \rceil = 2$ points.

Therefore, this hemisphere contains 4 points. Done.

Another variant

If

$$m_1 + m_2 + \cdots + m_n - n + 1$$

objects are placed in n boxes, then:

Some box i will contain $\geq m_i$ objects

proof: Assume box i has at most $m_i - 1$ objects

Total # objects $\leq \sum_{i=1}^{n} (m_i - 1) = m_1 + \cdots + m_n - n$,

a contradiction.

Example:

Given 7 objects and two boxes, either

box 1 has ≥ 3 objects or box 2 has ≥ 5 objects

$$m_1 = 3 \quad , \quad m_2 = 5 \quad , \quad n = 2$$

$$m_1 + m_2 - n + 1 = 7$$