Proof by Induction

- Proof technique
- Prove something of this form

$$\forall n \in \mathbb{N},\ P(n)$$

- Typically, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

Examples:

- Prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$

$$\forall n \in \mathbb{N},\ P(n) \quad \text{where } P(n): \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- Prove that every positive integer n can be written as

$$n = m2^k \quad \text{where } m \text{ is odd},\ k \in \mathbb{N}$$

- Prove that $\overbrace{1 + 3 + 5 + \cdots + (2n-1)}^{\text{first } n \text{ odds}} = n^2$ for all $n \geq 1$

How to prove by induction

To prove $P(n)$ is true for all $n \geq n_0$ (typically $n_0 = 0$)

- Base case : Prove $P(n_0)$ is true (verification)
- Inductive step: Prove $\forall\, k \geq n_0$, $P(k) \Longrightarrow P(k+1)$

  (Assume $P(k)$ is true, use that to prove $P(k+1)$ is true)

This establishes : $\forall\, n \geq n_0$, $P(n)$ is true

Example 1 : Prove $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ for all $n \geqslant 0$

Base case: $n_0 = 0$: $P(0)$: $\sum_{i=0}^{0} 2^i = 2^{0+1} - 1$

$$2^0 = 2^1 - 1$$

$$1 = 1 \checkmark$$

Inductive step: $\forall\, k \geqslant 0,\ P(k) \Rightarrow P(k+1)$

$P(k)$: $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$ (Inductive hypothesis)

$P(k+1)$: $\sum_{i=0}^{k+1} 2^i = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$

$$\sum_{i=0}^{k+1} 2^i = \sum_{i=0}^{k} 2^i + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$$

$$= 2^{k+1} + 2^{k+1} - 1 = 2 \cdot 2^{k+1} - 1$$

$$= 2^{k+2} - 1 \text{. Done}$$

Example 2

$$\prod_{i=1}^{n}\left(1+\frac{1}{i}\right) = n+1 \quad \text{for all} \quad n \geq 1.$$

Base case: $n_0 = 1.$ $\prod_{i=1}^{1}\left(1+\frac{1}{i}\right) = 1+1$

$$1+\frac{1}{1} = 1+1 \quad ✓$$

Inductive step: $\forall\, k \geq 1,\ P(k) \implies P(k+1)$

$P(k)$: $\prod_{i=1}^{k}\left(1+\frac{1}{i}\right) = k+1$ (Inductive hypo)

$P(k+1)$: $\prod_{i=1}^{k+1}\left(1+\frac{1}{i}\right) = (k+1)+1 = k+2$

$$\prod_{i=1}^{k+1}\left(1+\frac{1}{i}\right) = \prod_{i=1}^{k}\left(1+\frac{1}{i}\right) \times \left(1+\frac{1}{k+1}\right) = (k+1)\left(1+\frac{1}{k+1}\right) = (k+1)+1$$

$$= k+2.$$

Example 3

$$T_0 = 0$$

$$T_{n+1} = T_n \overline{T_n} \qquad (\overline{T_n} = T_n \text{ with bits negated})$$

$T_1 = T_0 \overline{T_0} = 01$

$T_2 = T_1 \overline{T_1} = 0110$

$T_3 = T_2 \overline{T_2} = 01101001$

Prove T_n has 2^n bits for all $n \in \mathbb{N} = \{0, 1, 2, \dots\}$

<u>Base case:</u> $n_0 = 0$. $P(0)$: T_0 has 2^0 bits ✓

<u>Inductive step:</u> $\forall k \geq 0,\ P(k) \Longrightarrow P(k+1)$

$P(k)$: T_k has 2^k bits (Ind. hyp)

$P(k+1)$: T_{k+1} has 2^{k+1} bits

$T_{k+1} = T_k \overline{T_k}$ which has $2^k + 2^k$ bits $= 2 \cdot 2^k$ bits $= 2^{k+1}$ bits.

Done.

Example4: Prove that T_n starts with 01 for all $n \geq 1$

Base case: $n_0 = 1$. $T_1 = 01$ so it starts with 01 ✓

Inductive step: $\forall k \geq 1, \; P(k) \Rightarrow P(k+1)$

$P(k)$: T_k starts with 01 (Inductive hyp)

$P(k+1)$: T_{k+1} starts with 01

$T_{k+1} = T_k \overline{T_k}$ and therefore T_{k+1} starts with 01 since

T_k starts with 01. Done.

Example 5. Prove T_n ends in 10 if n even and 01 if n odd for $n \geq 1$

Base case: $n_0 = 1$. $T_1 = 01$, it ends in 01 and 1 is odd ✓

Inductive step: $\forall\, k \geq 1$, $P(k) \Rightarrow P(k+1)$

$$T_{k+1} = T_k \overline{T_k}$$

$k+1$ even $\Rightarrow$ k is odd $\Rightarrow$ T_k ends in 01 (Inductive hyp.)

$\Rightarrow$ $\overline{T_k}$ ends in 10 $\Rightarrow$ T_{k+1} ends in 10.

$k+1$ is odd $\Rightarrow$ ... (symmetric argument)

Example 6

Prove $n^3 - n$ is a multiple of 3 for all $n \geq 0$

Base case: $n_0 = 0$. $P(0) : 0^3 - 0 = 0$ is multiple of 3 ✓

Inductive step: $\forall k \geq 0,\ P(k) \Longrightarrow P(k+1)$

$P(k) : k^3 - k = 3m$ (Inductive hypothesis)

$P(k+1) : (k+1)^3 - (k+1) = 3m'$ $\qquad m, m' \in \mathbb{Z}$

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + \cancel{1} - k - \cancel{1}$$

$$= (k^3 - k) + 3(k^2 + k)$$

Ind. hyp.

$$= 3m + 3(k^2 + k) = 3\underbrace{(m + k^2 + k)}_{m'}$$

. Done