What's wrong with this:

$$\text{Prove } \sum_{i=1}^{n} i = \frac{n^2+n+\sqrt{\pi}}{2}, \quad \forall n \in \mathbb{N}$$

$$P(k): \sum_{i=1}^{k} i = \frac{k^2+k+\sqrt{\pi}}{2} \qquad \text{(Inductive hypothesis)}$$

$$P(k+1): \sum_{i=1}^{k+1} i = \frac{(k+1)^2+(k+1)+\sqrt{\pi}}{2}$$

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \frac{k^2+k+\sqrt{\pi}}{2} + \frac{2(k+1)}{2} = \frac{(k+1)^2+(k+1)+\sqrt{\pi}}{2}$$

??? :(

$\forall n \geq 2$. n lines no two of which are // intersect in one point.

Base case: $n_0 = 2$. $P(2)$: 2 lines not // intersect in one point ✓

Inductive step: $\forall k \geq 2$. $P(k) \Longrightarrow P(k+1)$

Given $k+1$ lines $\ell_1, \ell_2, \ell_3, \ldots, \ell_k, \ell_{k+1}$ no two of which are //,

Consider the two sets of lines

- $\ell_1, \ell_2, \ell_3, \ldots, \ell_{k-1}, \ell_{k+1}$ (no two are //)
- $\ell_1, \ell_2, \ell_3, \ldots, \ell_{k-1}, \ell_k$ (no two are //)

Each set has k lines $\Longrightarrow$ all lines in each set intersect in one point, namely the intersection of ℓ_1 & ℓ_2.

Therefore all $k+1$ lines go through that point !!!!

What was wrong?

Proof does not work for $k=2$ try

$$\ell_1, \ell_2, \ell_3$$

drop one line

ℓ_1, ℓ_2 $\qquad\qquad$ $\underbrace{\ell_1, \ell_3}$

ℓ_2 is not here!

Here's another of those:

For every $n \geqslant 12$, $n = 3x + 7y$ where $x, y \in \mathbb{N}$.

Base case: $n_0 = 12$. $P(12)$: $12 = 3 \times 4 + 7 \times 0$ ✓

Inductive step: $\forall k \geqslant 12.\ P(k) \Rightarrow P(k+1)$

$P(k)$: $k = 3x + 7y$, $x, y \in \mathbb{N}$

$P(k+1)$: $k+1 = 3x' + 7y'$, $x', y' \in \mathbb{N}$

$$k+1 = 3x + 7y + 1 = 3x + 7y + \underbrace{7 - 2\times 3}$$

$$= 3(x-2) + 7(y+1)$$

$\in \mathbb{N}$?

Strong Induction

Base case Prove $P(k)$ true for $k \leq n_0$

Inductive Step: $\forall k \geq n_0 . \underbrace{\bigwedge_{i \leq k} P(i)}_{\text{Inductive hypo.}} \Rightarrow P(k+1)$

In other words, assume the property is true **up to** k, then prove it's also true for $k+1$.

- Note: It's typical that we won't need all statements up to P_k to be true, but only some of them.

The notation $\bigwedge_{i \leq k} P(i)$ means $P(k) \wedge P(k-1) \wedge P(k-2) \wedge \ldots$

Example 1: For every $n \geq 12$, $n = 3x + 7y$ where $x, y \in \mathbb{N}$

Base case: $P(12): 12 = 3(4) + 7(0)$ ✓

$P(13): 13 = 3(2) + 7(1)$ ✓

$P(14): 14 = 3(0) + 7(2)$ ✓

Inductive hyp. $\bigwedge_{12 \leq i \leq k} P(i)$: $i = 3x + 7y$ for all $12 \leq i \leq k$

Inductive step: $\forall k \geq n_0, \bigwedge_{12 \leq i \leq k} P(k) \Rightarrow P(k+1)$. $P(k+1): k+1 = 3x + 7y$

$$k+1 = (k+1) - 3 + 3 = \underbrace{k-2} + 3$$

$$= 3x' + 7y' + 3$$

$$= 3(x'+1) + 7y'$$

$$= 3x + 7y$$

Proof works when $k-2 \geq 12 \Rightarrow k \geq 14$. So $n_0 = 14$.

Example 2. Every $n \geqslant 1$ can be written as $n = m \cdot 2^i$

where m is odd, $i \geqslant 0$

Base case: $1 = 1 \cdot 2^0$ ✓

Inductive hyp. $\bigwedge_{1 \leqslant j \leqslant k} P(j) : j = m \cdot 2^i \quad \forall \quad 1 \leqslant j \leqslant k$

Inductive step: $\forall k \geqslant n_0, \bigwedge_{1 \leqslant i \leqslant k} P(i) \Rightarrow P(k+1)$. $P(k+1) : k+1 = m \cdot 2^i$

$k+1$ odd : $k+1 = (k+1) 2^0$

$k+1$ even $\Rightarrow k+1 = 2j$ where $j < k$?

so $P(j)$ is true and $j = m \cdot 2^l$

Therefore, $k+1 = 2 [m \cdot 2^l] = m \cdot 2^{l+1} = m \cdot 2^i$.

Proof works as long as $\begin{cases} j \leqslant k \Rightarrow \frac{k+1}{2} \leqslant k \Rightarrow k+1 \leqslant 2k \Rightarrow k \geqslant 1. \\ j \geqslant 1 \Rightarrow \frac{k+1}{2} \geqslant 1 \Rightarrow k \geqslant 1. \end{cases}$

So $n_0 = 1$.