Strong induction & Recurrences

- Strong induction works well with recurrences; for instance, when we have something like

$$a_n = f(a_{n-1}, a_{n-2}, \dots)$$

recurrence

- In other words, a_n is expressed in terms of $a_{n-1}, a_{n-2}, \dots$

- Proving a property of a_n by strong induction can use

$$\bigwedge_{i \leq k} P(k) \implies P(k+1)$$

since $$a_{k+1} = f\big(\underbrace{a_k, a_{k-1}, \dots}_{\text{use the property here}}\big)$$

Example 3:

Consider

$$a_1 = 3$$
$$a_2 = 5$$
$$a_n = 3a_{n-1} - 2a_{n-2}, \quad n \geqslant 3$$

Let's find a few a_n's:

$$a_3 = 3a_2 - 2a_1 = 3\cdot 5 - 2\cdot 3 = 15 - 6 = 9$$
$$a_4 = 3a_3 - 2a_2 = 3\cdot 9 - 2\cdot 5 = 27 - 10 = 17$$
$$a_5 = 3a_4 - 2a_3 = 3\cdot 17 - 2\cdot 9 = 51 - 18 = 33$$
$$\vdots$$

Guess $a_n =$

3, 5, 9, 17, 33, 65, 129,

$$\boxed{a_n = 2^n + 1}$$

Prove $a_n = 2^n + 1$ for all $n \geq 1$

<u>Base case</u> $P(1)$: $a_1 = 2^1 + 1 = 3$ ✓

$P(2)$: $a_2 = 2^2 + 1 = 5$ ✓

$\vdots$

Inductive hypothesis : $\bigwedge_{1 \leq i \leq k} P(i)$: $a_i = 2^i + 1$ $\forall\ 1 \leq i \leq k$

<u>Inductive step:</u>

$\forall k \geq n_0, \bigwedge_{1 \leq i \leq k} P(k) \Rightarrow P(k+1)$. $P(k+1)$: $a_{k+1} = 2^{k+1} + 1$

$$\underbrace{a_{k+1} = 3a_k - 2a_{k-1}}_{\text{use recurrence}} = 3 \cdot (2^k + 1) - 2(2^{k-1} + 1)$$

$$= 3 \cdot 2^k - 2 \cdot 2^{k-1} + 1$$

$$= 3 \cdot 2^k - 2^k + 1$$

$$= 2 \cdot 2^k + 1$$

$$= 2^{k+1} + 1$$

Proof works if $\underbrace{k - 1 \geq 1}_{a_{k-1} \text{ defined}} \Rightarrow k \geq 2$. So $n_0 = 2$.

Example 4. Fibonacci Sequence

$$F_0 = 0$$

$$F_1 = 1$$

$$F_n = F_{n-1} + F_{n-2} \ , \ n \geq 2$$

Prove $F_n = \frac{1}{\sqrt{5}}\left[\phi^n - (1-\phi)^n\right]$ for $n \geq 0$

where $\phi = \frac{1+\sqrt{5}}{2}$ (ϕ is called the golden ratio)

Note: Both ϕ and $1-\phi$ are solutions to $\frac{1}{x} + \frac{1}{x^2} = 1$

Base case: $P(0)$: $0 = \frac{1}{\sqrt{5}}\left[\phi^0 - (1-\phi)^0\right] = \frac{1}{\sqrt{5}}[1-1] = 0$ ✓

$P(1)$: $1 = \frac{1}{\sqrt{5}}\left[\phi - (1-\phi)\right] = \frac{1}{\sqrt{5}}(2\phi - 1) = 1$ ✓

Inductive hypothesis: $\bigwedge\limits_{0\leq i\leq k} P(i)$: $F_i = \frac{1}{\sqrt{5}}\left[\phi^{i} - (1-\phi)^{i+1}\right]$ $\forall\ 0\leq i\leq k$

Inductive step:

$$\forall k \geq n_0,\ \bigwedge_{0\leq i\leq k} P(i) \Longrightarrow P(k+1)$$

$$P(k+1):\quad F_{k+1} = \frac{1}{\sqrt{5}}\left[\phi^{k+1} - (1-\phi)^{k+1}\right]$$

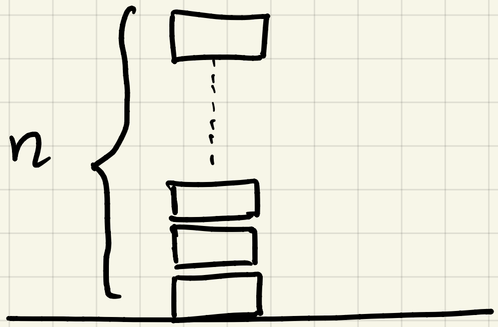
$$F_{k+1} = F_k + F_{k-1} = \frac{1}{\sqrt{5}}\left[\phi^{k} - (1-\phi)^{k}\right] + \frac{1}{\sqrt{5}}\left[\phi^{k-1} + (1-\phi)^{k-1}\right]$$

$$= \frac{1}{\sqrt{5}}\phi^{k+1}\underbrace{\left[\frac{1}{\phi}+\frac{1}{\phi^2}\right]}_{1} - \frac{1}{\sqrt{5}}(1-\phi)^{k+1}\underbrace{\left[\frac{1}{1-\phi}+\frac{1}{(1-\phi)^2}\right]}_{1}$$

$$= \frac{1}{\sqrt{5}}\left[\phi^{k+1} - (1-\phi)^{k+1}\right]$$

Proof works if $\underbrace{k-1 \geq 0}_{F_{k-1}\text{ defined}} \Rightarrow k \geq 1$. So $n_0 = 1$.

Consider a game with n blocks stacked in a tower



The goal is to split the stack repeatedly until we have n stacks of height 1.

Prove that for all $n \geq 1$, we need exactly $n-1$ splits.

Base case: $P(1)$: A stack of 1 block needs $1-1=0$ splits ✓

Inductive hypothesis: $\bigwedge_{1\leq i\leq k} P(i)$: stack of i blocks needs $i-1$ splits $\forall\, 1\leq i\leq k$

Inductive step: $\forall\, k \geq n_0,\ \bigwedge_{1\leq i\leq k} P(k) \Rightarrow P(k+1)$

$P(k+1)$: A stack of $k+1$ blocks requires k splits.

Make a move: First split makes two stacks of size s and $k+1-s$

$$1 \leqslant s \leqslant k \quad \text{and} \quad 1 \leqslant k+1-s \leqslant k$$

So $P(s)$ and $P(k+1-s)$ are true.

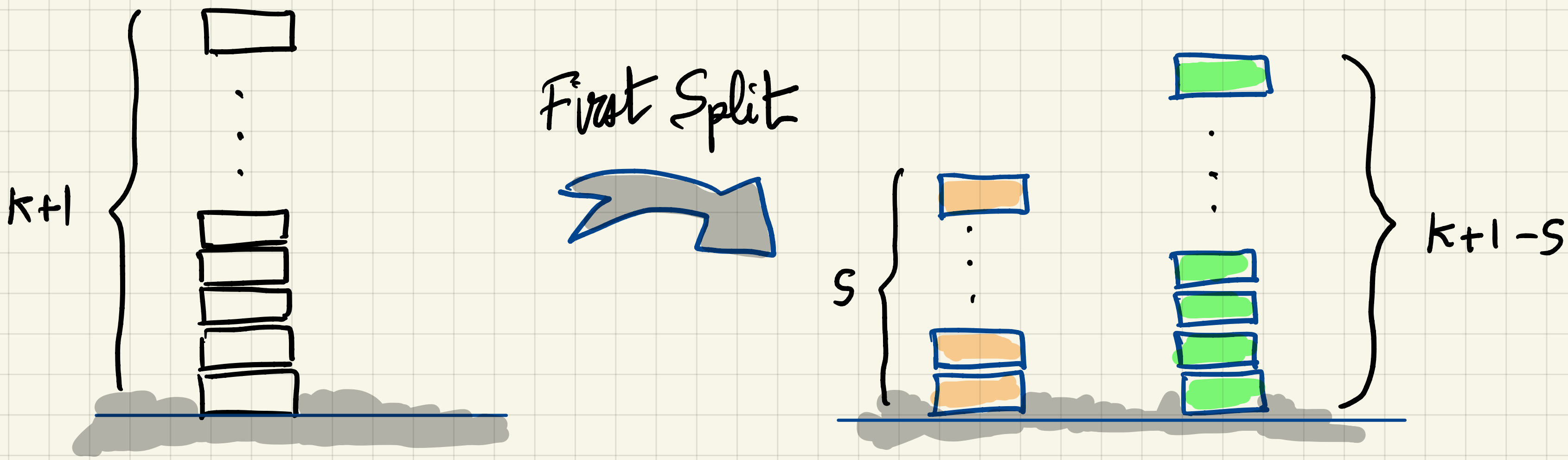
Total number of splits $= 1 + (s-1) + (k+1-s-1)$

$= 1 + s - 1 + k + 1 - s - 1$

$= k.$

Proof works if $k+1 \geqslant 2$ (first split exists.) $\Rightarrow k \geqslant 1$

$n_0 = 1$ ✓

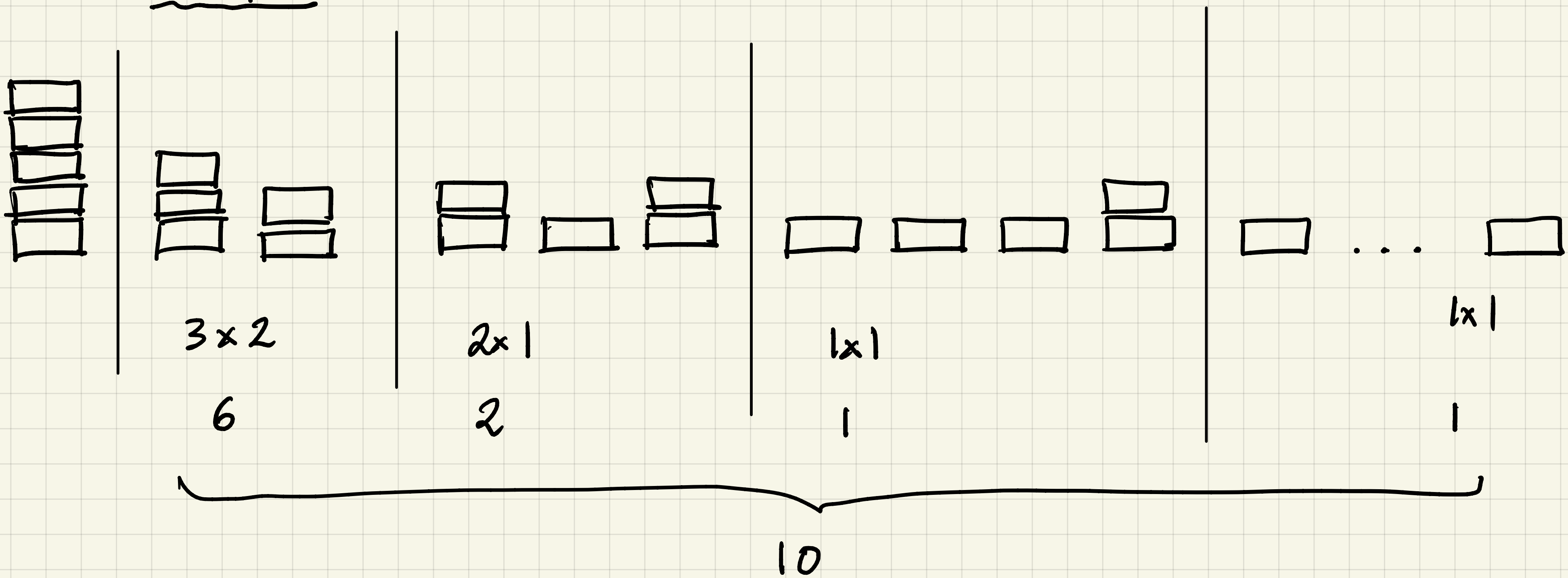


$$1 \leqslant s \leqslant k$$

$$1 \leqslant k+1-s \leqslant k$$

__Variation:__ Assume that if you split n into s and $n-s$ you receive a score of $n(n-s)$.

__Example:__



Prove that the score is always $\binom{n}{2}$

Base case: $P(1)$: $\binom{1}{2} = 0$ ✓

Inductive hyp: $\bigwedge_{1 \leq i \leq k} P(i)$: i blocks have score $\binom{i}{2}$ $\forall\ 1 \leq i \leq k$

Inductive step: $\forall k \geq n_0, \bigwedge_{1 \leq i \leq k} P(i) \implies P(k+1)$

$P(k+1)$: score is $\binom{k+1}{2}$ for $k+1$ blocks

For $k+1$, the score is

$$\underbrace{s(k+1-s)}_{\text{First split}} + \binom{s}{2} + \binom{k+1-s}{2}$$

$$= s(k+1-s) + \frac{s(s-1)}{2} + \frac{(k+1-s)(k-s)}{2}$$

$$\vdots$$

$$= \frac{k(k+1)}{2} = \binom{k+1}{2}$$