Solving Recurrences

Consider the Fibonacci sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

$$F_0 = 0$$
$$F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2} \qquad n \geq 2$$

We proved by induction (strong induction)

$$F_n = \frac{1}{\sqrt{5}}\left[\phi^n - (1-\phi)^n\right], \quad \underbrace{\phi = \frac{1+\sqrt{5}}{2}} = 1.618\ldots$$

$$\approx \frac{1}{\sqrt{5}}\phi^n$$

(why?)

the golden ratio

It's been observed that ratio of consecutive Fib. numbers converges to ϕ

$$\frac{1}{0}, \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \cdots \longrightarrow \phi \approx 1.618\ldots$$

[wishful thinking] what if $F_n = c\,p^n$

$$\frac{F_{n+1}}{F_n} = \frac{c\,p^{n+1}}{c\,p^n} = p \quad (\text{make } p = \phi)$$

Does this work?

$$F_n = F_{n-1} + F_{n-2}$$
$$cp^n = cp^{n-1} + cp^{n-2}$$
$$p^n = p^{n-1} + p^{n-2}$$
$$p^2 = p + 1$$

ϕ is a solution to this

This would work if p is a solution to $x^2 = x+1 \quad (x^2 - x - 1 = 0)$

Problem: Can't make $F_0 = cp^0$ and $F_1 = cp^1$

but $x^2 - x - 1$ has two solutions $\nearrow p = \phi$, $\searrow q = 1-\phi$

make $F_n = c_1 p^n + c_2 q^n$

$$F_0 = c_1 p^0 + c_2 q^0 = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

substitute

$$F_1 = c_1 p + c_2 q = c_1 \phi + c_2 (1-\phi) = 1$$

$$c_1 \phi - c_1 (1-\phi) = 1$$

$$c_1 (2\phi - 1) = 1$$

$$c_1 \sqrt{5} = 1$$

$$c_1 = 1/\sqrt{5} \Rightarrow c_2 = -\frac{1}{\sqrt{5}}$$

$$F_n = \frac{1}{\sqrt{5}} \left[\phi^n - (1-\phi)^n \right].$$

Let's prove $F_n = c_1 p^n + c_2 q^n$ satisfies the recurrence

$$F_n = F_{n-1} + F_{n-2}$$

$$c_1 p^n + c_2 q^n = c_1 p^{n-1} + c_2 q^{n-1} + c_1 p^{n-2} + c_2 q^{n-2}$$

$$c_1 p^n + c_2 q^n = c_1 [p^{n-1} + p^{n-2}] + c_2 [q^{n-1} + q^{n-2}]$$

Since p and q are solutions to $x^2 - x - 1 = 0$

$$p^2 = p + 1 \qquad q^2 = q + 1$$

$$p^n = p^{n-1} + p^{n-2} \qquad q^n = q^{n-1} + q^{n-2}$$

$$c_1 p^n + c_2 q^n = c_1 p^n + c_2 q^n$$

In general

characteristic equation

$$a_n = A a_{n-1} + B a_{n-2}$$

$$x^2 = Ax + B \quad p, q \text{ solutions.}$$

$$a_n = \begin{cases} c_1 p^n + c_2 q^n & p \neq q \\ c_1 p^n + c_2 n p^n & p = q \end{cases}$$

we can prove the above fact using strong induction.

Example: $a_0 = 0$

$$a_n = 2a_{n-1} + 1 \qquad n \geq 1$$

$0, 1, 3, 7, 15, 31, \ldots$

$$\underbrace{2^n - 1}_{a_n} = 2\underbrace{\left[2^{n-1} - 1\right]}_{a_{n-1}} + 1$$

Guess: $a_n = 2^n - 1$

How do we make sure:
- make sure it satisfies recurrence
- good for first few terms

OR

prove it by induction

proof by induction:

Base Case $a_0 = 2^0 - 1 = 1 - 1 = 0$ ✓

Inductive step: $\forall k \geq 0,\ P(k) \Rightarrow P(k+1)$

$P(k)$: $a_k = 2^k - 1$

$P(k+1)$: $a_{k+1} = 2^{k+1} - 1$

$$a_{k+1} = 2a_k + 1 = 2\left[2^k - 1\right] + 1 = 2^{k+1} - 2 + 1 = 2^{k+1} - 1.$$

Avoid guessing:

Make recurrence with desired form

$$a_n = 2a_{n-1} + 1$$ bad

$$a_{n-1} = 2a_{n-2} + 1$$

$$(-)\quad a_n - a_{n-1} = 2a_{n-1} - 2a_{n-2} + (1-1)$$

$$a_n = 3a_{n-1} - 2a_{n-2} \qquad n \geq 2$$

$$x^2 = 3x - 2 \quad p = 2, \quad q = 1$$

$$a_n = c_1 2^n + c_2 1^n = c_1 2^n + c_2$$

$$a_0 = c_1 2^0 + c_2 = c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$$

$$a_1 = c_1 2 + c_2 = 2c_1 - c_1 = c_1 = 1$$

$$a_n = 2^n - 1$$

Example:

$$a_1 = 0 \qquad a_2 = 6$$

$$a_n = -a_{n-1} + 3 \times 2^{n-1} \qquad n \geq 2$$

$$2a_{n-1} = -2a_{n-2} + 3 \times 2^{n-2} \times 2$$

$$a_n - 2a_{n-1} = -a_{n-1} + 2a_{n-2}$$

$$a_n = a_{n-1} + 2a_{n-2} \qquad n \geq 3$$

$$x^2 = x + 2 \rightarrow p = 2, \quad q = -1$$

$$a_n = c_1 2^n + c_2 (-1)^n$$

$$a_1 = 2c_1 - c_2 = 0$$

$$a_2 = 4c_1 + c_2 = 6 \qquad (+)$$

$$6c_1 = 6 \rightarrow c_1 = 1 \Rightarrow c_2 = 2$$

$$a_n = 2^n + 2(-1)^n$$

Example:

$a_0 = 0 \qquad a_1 = 2$

$$a_n = 4a_{n-1} - 4a_{n-2} \qquad n \geq 2$$

$$x^2 = 4x - 4$$

$$x^2 - 4x + 4 = 0$$

$$(x-2)^2 = 0 \quad \begin{cases} p = 2 \\ q = 2 \end{cases}$$

$$a_n = c_1 2^n + c_2 n . 2^n \qquad (p=q)$$

$$a_0 = c_1 = 0$$

$$a_1 = c_2 . 1 . 2^1 = 2c_2 = 2 \Rightarrow c_2 = 1$$

$$a_n = n 2^n$$

Linear Homogeneous Recurrence

$$a_n = \sum_{i=1}^{k} \beta_i a_{n-i}$$

$$x^k = \sum_{i=1}^{k} \beta_i x^{k-i}$$

Example: $a_n = Aa_{n-1} + Ba_{n-2} + Ca_{n-3}$

$$x^3 = Ax^2 + Bx + C \quad \begin{array}{l} \nearrow p \\ \rightarrow q \\ \searrow r \end{array}$$

p, q, r all different : $a_n = c_1 p^n + c_2 q^n + c_3 r^n$

$p \neq q = r$: $a_n = c_1 p^n + c_2 q^n + c_3 n q^n$

$p = q = r$: $a_n = c_1 p^n + c_2 n p^n + c_3 n^2 p^n$

Solve for c_1, c_2, and c_3 using a_0, a_1, and a_2.