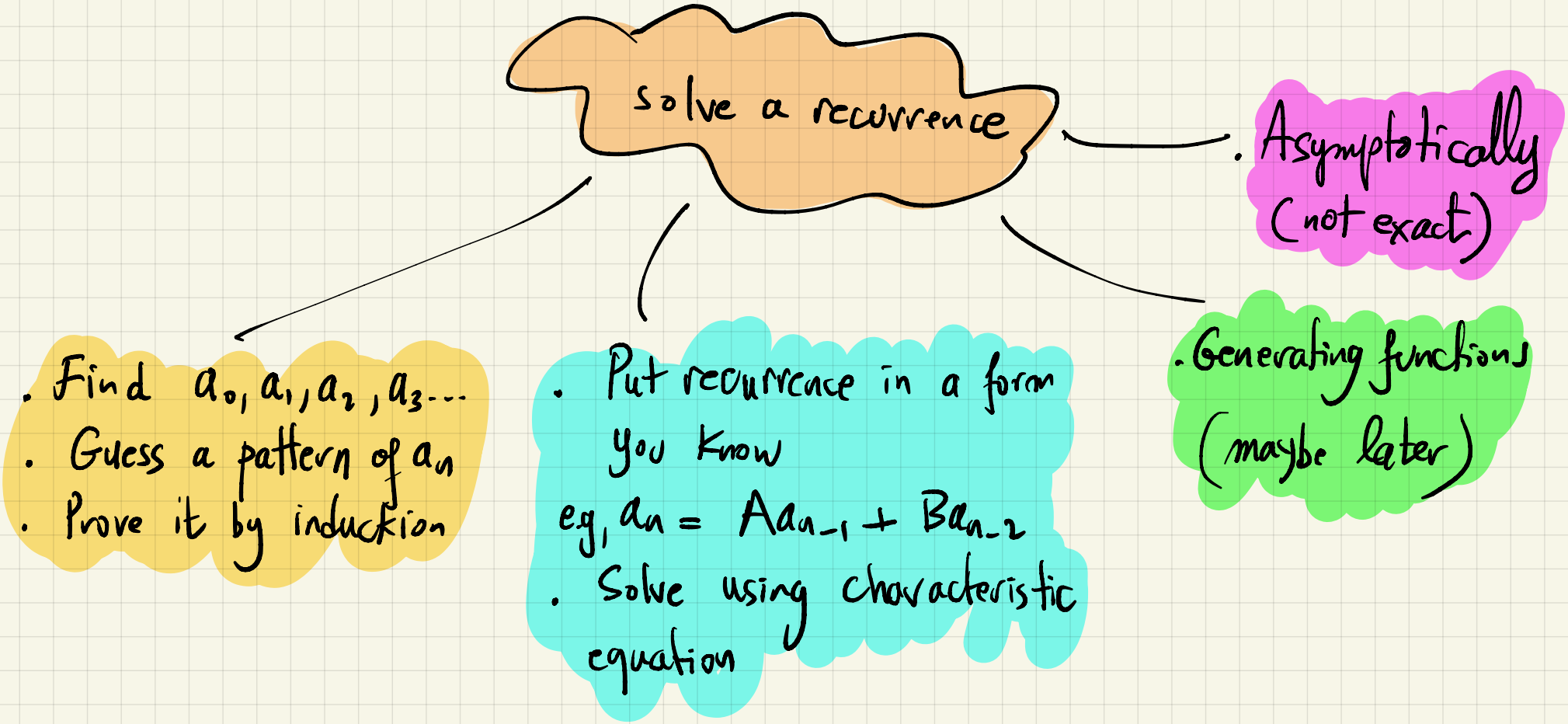
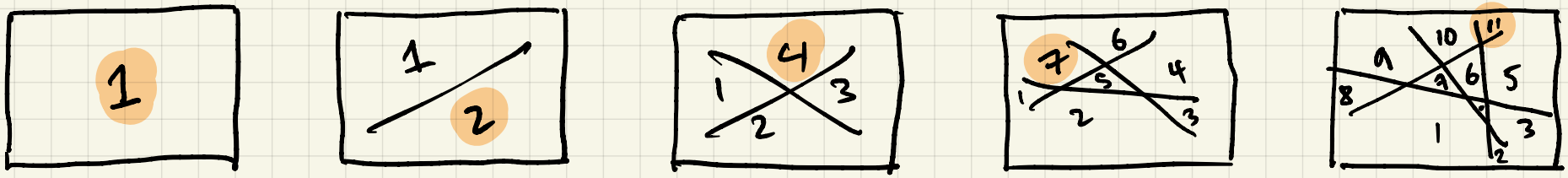
Today: How recurrences help with counting

- Instead of counting the exact thing
  Find a recurrence for it
- At the price of having to solve recurrence



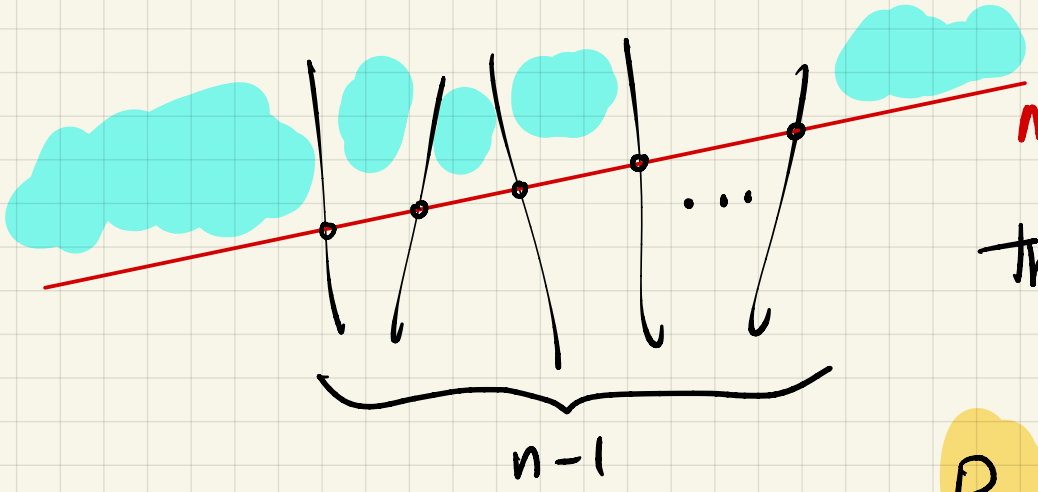
Example 1: How many regions n lines make in the plane if no two are parallel and no three intersect in one point



0 lines | 1 line | 2 lines | 3 lines | 4 lines

let R_n = # regions made by n lines (Name It!!)

$R_0 = 1$, $R_1 = 2$, $R_2 = 4$, $R_3 = 7$, $R_4 = 11$



n^{th} line

the n^{th} line adds n new regions

$$R_n = R_{n-1} + n$$

n	0	1	2	3	4	5	6	...
	1	2	4	7	11	16	22	...

Example: $11 = 4 + 7$

$3 + 4$

$2 + 2$

$1 + 1$

$$R_n = (1 + 2 + \cdots + n) + 1$$

$$R_n = \frac{n(n+1)}{2} + 1$$

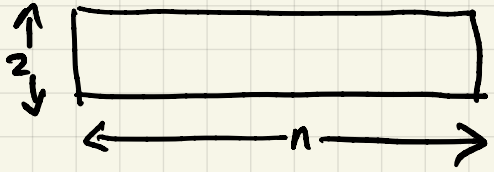
Prove it by induction.

Base case: $R_0 = \frac{0(0+1)}{2} + 1 = 1$ ✓

Inductive step: $\forall\, k \geq 1,\ P(k) \Rightarrow P(k+1)$

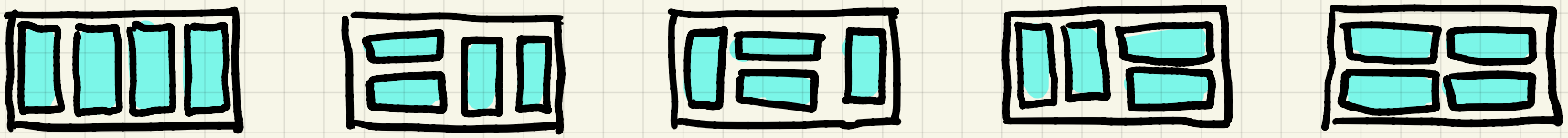
$$R_{k+1} = \underbrace{R_k}_{\text{ind. hypo.}} + (k+1) = \underbrace{\frac{k(k+1)}{2} + 1}_{\text{ind. hypo.}} + k + 1 = \cdots = \frac{(k+1)(k+2)}{2} + 1.$$

Example 2. Tiling a 2xn rectangle by dominos.



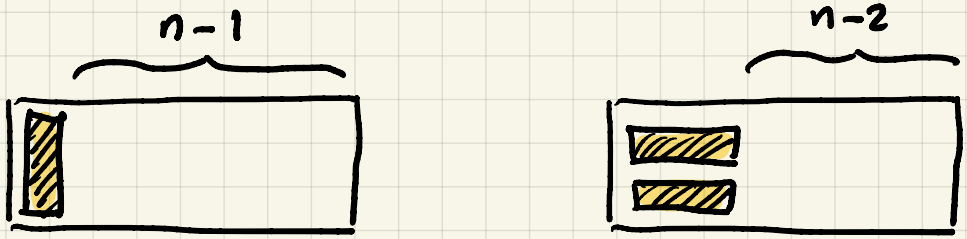
In how many ways can we tile a 2xn rectangle with dominos

n=4:



Let a_n be # ways we can tile a rectangle of length n ($a_4 = 5$)

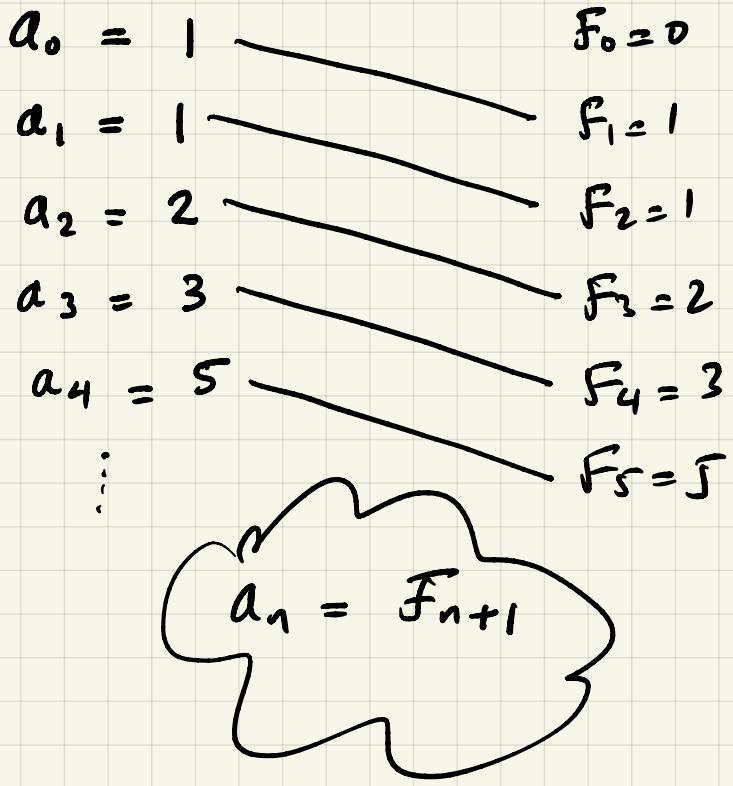
Two cases



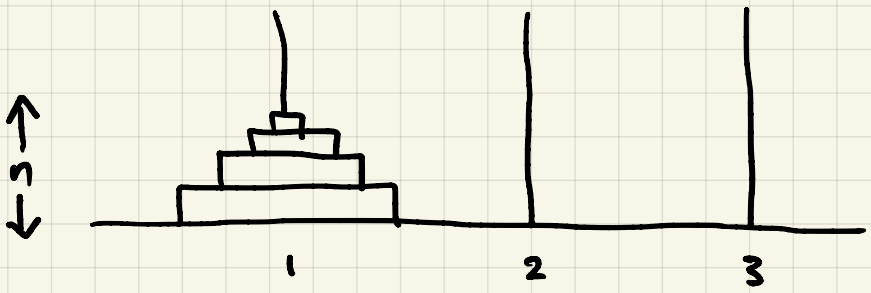
a_{n-1} ways of finishing

a_{n-2} ways of finishing

$$a_n = a_{n-1} + a_{n-2}$$ (why?)


$a_0 = 1$
$a_1 = 1$
$a_2 = 2$
$a_3 = 3$
$a_4 = 5$
$F_0 = 0$
$F_1 = 1$
$F_2 = 1$
$F_3 = 2$
$F_4 = 3$
$F_5 = 5$
$a_n = F_{n+1}$

Example 3: Tower of Hanoi

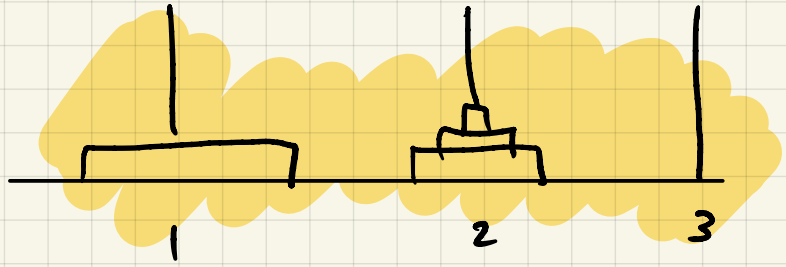


Move n disks from peg 1 to peg 3
- move one disk at a time
- no disk can sit on top of a smaller one

How many moves are needed to transfer a pile of n disks ?

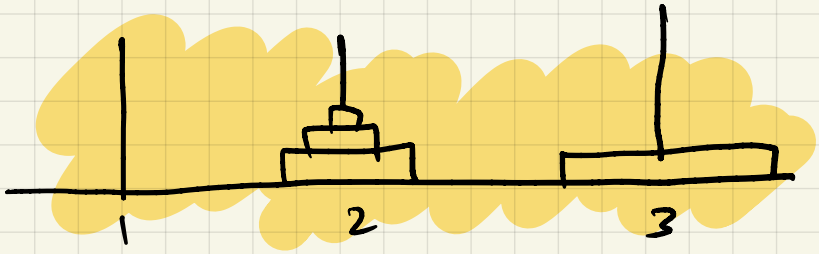
Let a_n = # moves to transfer n disks.

In order to move the largest disk, we must reach this:



I must have moved n-1 disks with a_{n-1} moves

Then make 1 move



Then I need another a_{n-1} moves

$$a_n = a_{n-1} + 1 + a_{n-1}$$

$$a_n = 2a_{n-1} + 1$$

- We solved this recurrence by eliminating 1 and putting it in the form $a_n = Aa_{n-1} + Ba_{n-2}$
- Another way: Establish a pattern

$$a_0 = 0 \quad a_1 = 1 \quad a_2 = 3 \quad a_3 = 7 \quad a_4 = 15 \quad a_5 = 31 \quad \ldots$$

$$a_n = 2^n - 1$$