Generating Functions

The generating function of the sequence

$$a_0, \ a_1, \ a_2, \ a_3, \ \dots$$

is $f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \dots$

$$= \sum_{n=0}^{\infty} a_n x^n$$

Example: Generating function for Fibonacci

$$f(x) = 0 \cdot x^0 + 1 \cdot x^1 + 1 \cdot x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \dots$$

The n^{th} derivative of $f(x)$ at $x=0$ divided by $n!$ is a_n

$$a_n = \frac{f^{(n)}(0)}{n!} \quad \text{(for any sequence)}$$

$$f^{(0)}(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots$$

$$f^{(0)}(0) = a_0 \Rightarrow \frac{f(0)}{0!} = a_0$$

$$f^{(1)}(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots$$

$$f^{(1)}(0) = a_1 \Rightarrow \frac{f^{(1)}(0)}{1!} = a_1$$

$$f^{(2)}(x) = 2a_2 + 6a_3x + \cdots$$

$$f^{(2)}(0) = 2a_2 \Rightarrow \frac{f^{(2)}(0)}{2!} = a_2$$

$$f^{(3)}(x) = 6a_3 + \cdots$$

$$f^{(3)}(0) = 6a_3 \Rightarrow \frac{f^{(3)}(0)}{3!} = a_3$$

$$\vdots$$

So $$f(x) = \sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(0)}{n!}}_{a_n} x^n$$ (Maclaurin series)

But what is $f(x)$ really ?

Finding it will help find a_n

Example: $a_1 = 0$; $a_n = 3 \times 2^{n-1} - a_{n-1}$, $n \geq 2$

$a_1 = 0$, $a_2 = 3 \times 2^{2-1} - a_1 = 6$, ...

Solution: $a_n = 2^n + 2(-1)^n$. [look for it]

Generating function:

$$f(x) = a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_1 x^1 + (3 \times 2 - a_1)\, x^2 + (3 \times 2^2 - a_2)\, x^3 + (3 \times 2^3 - a_3)\, x^4 + \cdots$$

$$= a_1 x + 6x^2(1 + 2x + 4x^2 + \cdots) - (a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots)$$

$$\boxed{f(x)} = 6x^2 \cdot \frac{1}{1-2x} - x\boxed{f(x)}$$

$$\left\{ 1 + a + a^2 + a^3 + \cdots = \frac{1}{1-a} \quad \text{if } |a| < 1 \right\}$$

$$f(x) = \frac{6x^2}{(1+x)(1-2x)}$$

$$f(x) = 2x^2\left[\frac{1}{1+x} + \frac{2}{1-2x}\right]$$

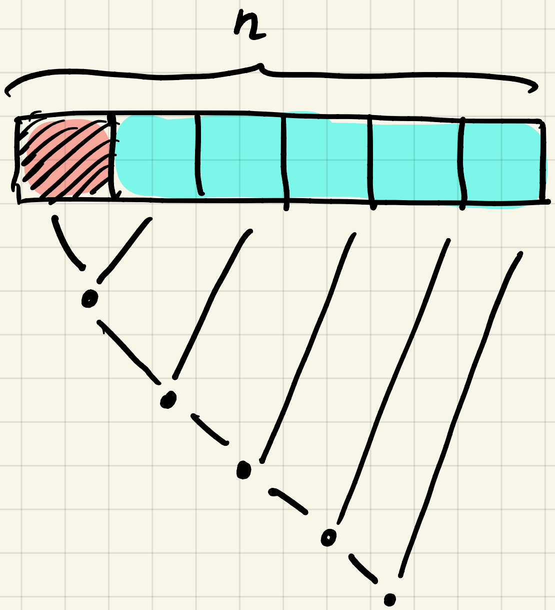
x^{n-2} $\qquad$ $2^{n-2}x^{n-2}$

$$f(x) = 2x^2[1 - x + x^2 - x^3 + \cdots] + 4x^2[1 + 2x + 4x^2 + \cdots]$$

Coefficient of x^n is $\underbrace{4 \cdot 2^{n-2} + 2(-1)^n}_{a_n}$

$$a_n = 2^n + 2(-1)^n$$

Sorting an array



(n−1) comparisions.

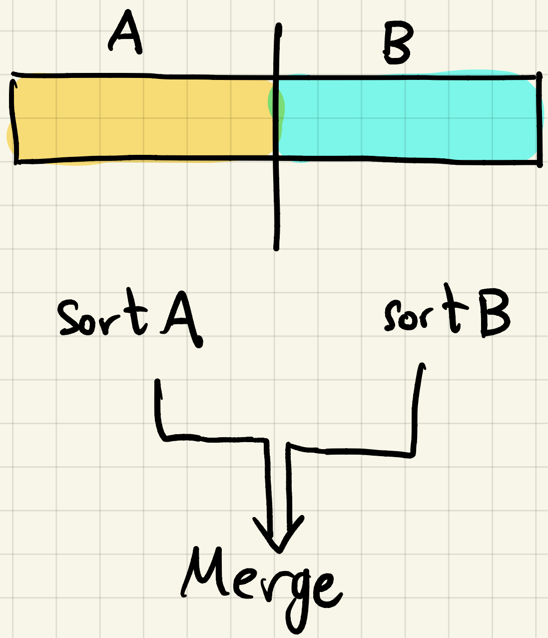
Repeatedly find the smallest element, move it to the beginning of array.

total: $(n-1) + (n-2) + (n-3) + \cdots + 1 = \dfrac{n(n-1)}{2} \approx \dfrac{n^2}{2}$

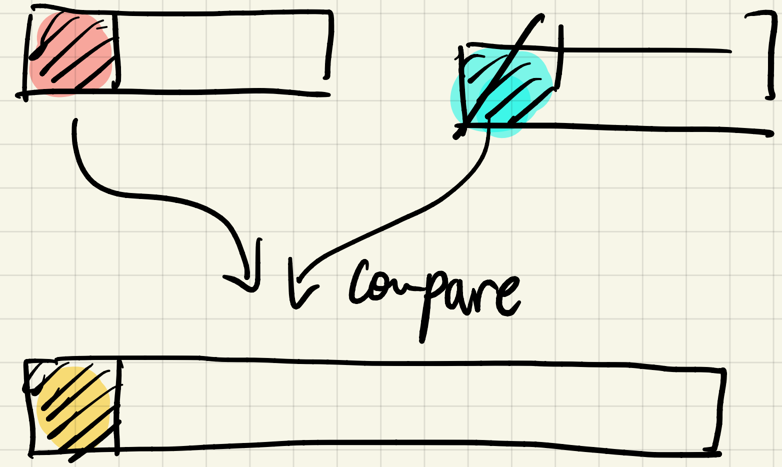
$$T_n = (n-1) + T_{n-1}, \quad T_1 = 0$$

A better idea

Merge Sort : Divide array into two "equal" size arrays



(Merging ...)



move smaller of the two

(n−1) comparisons.

[each comparison move an elem]

$$T_n = (n-1) + T_{\lfloor n/2 \rfloor} + T_{\lceil n/2 \rceil}$$

Approximation: $T_n = n + 2T_{n/2}$

(Assume n is a power of 2) $\quad T_1 = 0$

$T_1 \quad T_2 \quad T_4 \quad T_8 \quad T_{16}$

$a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4$

$a_n = T_{2^n}$

$$a_n = T_{2^n} = 2^n + 2T_{2^n/2} = 2^n + 2T_{2^{n-1}}$$

$$\boxed{a_n = 2^n + 2a_{n-1}} \qquad a_0 = 0$$

We have seen this before

solution: $a_n = n2^n$

$$a_n = T_{2^n} \iff T_n = a_{\log_2 n} = \log_2 n \cdot 2^{\log_2 n} = n \log_2 n .$$

Divide-and-Conquer algorithms have the following form of recurrences

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

See it in Algorithms course!