Sums & Products

- $\displaystyle\sum_{i=a}^{b} f(i) = f(a) + f(a+1) + \ldots + f(b)$

- $\displaystyle\prod_{i=a}^{b} f(i) = f(a) \cdot f(a+1) \cdot \ldots \cdot f(b)$

- $\displaystyle\sum_{i} c f(i) = c \sum_{i} f(i)$ (distributive)

- $\displaystyle\prod_{i=a}^{b} c f(i) = c^{b-a+1} \prod_{i=a}^{b} f(i)$ $(b \geq a)$

- $$\sum_i [f(i) + g(i)] = \sum_i f(i) + \sum_i g(i)$$

- $$\prod_i f(i) \cdot g(i) = \prod_i f(i) \cdot \prod_i g(i)$$

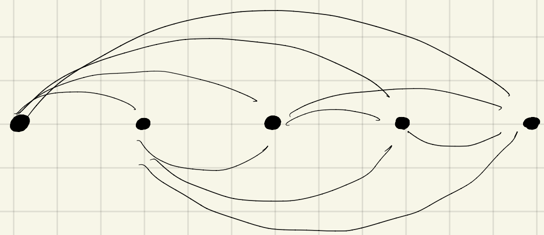
Example: $$\sum_{i=1}^{n} (3i - 5) = \sum_{i=1}^{n} 3i - \sum_{i=1}^{n} 5$$

splitting sum

$$= 3 \sum_{i=1}^{n} i - \sum_{i=1}^{n} 5$$

$$= \underbrace{3 \frac{n(n+1)}{2} - 5n}_{\text{combine}}$$

Counting pairs in nested loops



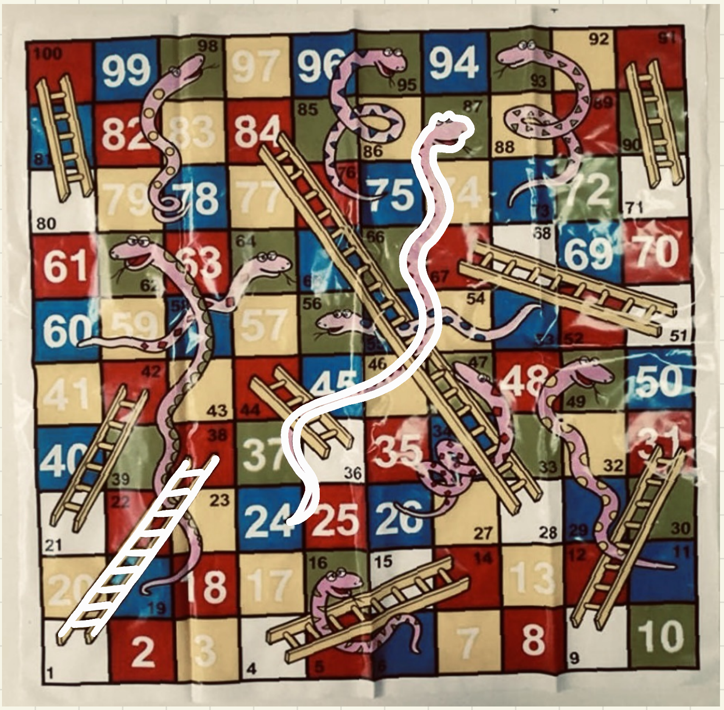
```
s = 0
for i in range(1, n+1):
    for j in range(i+1, n+1):
        s = s + 1
```

$$\sum_{i=1}^{n} \underbrace{\sum_{j=i+1}^{n} 1}_{n-(i+1)+1 \;=\; n-i}$$

(nested sum)

$$\sum_{i=1}^{n} (n-i) = \sum_{i=1}^{n} n - \sum_{i=1}^{n} i = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2} = \binom{n}{2}$$

Snakes & Ladders



snake : head $>$ tail

pros: No thinking required

cons: Placement of snakes & ladders so that it's not boring.

In how many ways can I place one snake on a board with n squares?

Plan: Three ways to count snakes

1) Addition rule
2) Product rule
3) Abstraction

A snake is just a pair of squares. In other words, a snakes corresponds to a pair of squares, and a pair of squares correspond to a snake.

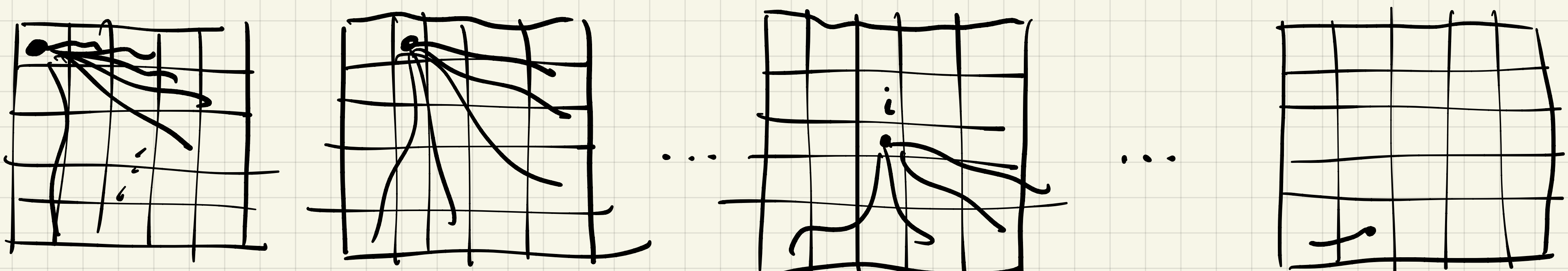
There are $\binom{n}{2} = \frac{n(n-1)}{2}$ pairs

Example : $n = 6$

5   4   3   2   1

snakes : $5+4+3+2+1 = \frac{5 \times 6}{2} = 15$

General n



$n-1$ $n-2$ $i-1$ 1

$$(n-1) + (n-2) + \cdots + 1 = \frac{(n-1)n}{2} = \binom{n}{2}$$

```
s = 0
for head in range(1, n+1):
    for tail in range(1, head):
        s ← s+1
```

$$\sum_{i=1}^{n}\sum_{j=1}^{i-1} 1 = \text{try it yourself}$$

$$\sum_{i=1}^{n}(i-1) = \sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = \frac{n(n+1)}{2} - n = \frac{n(n-1)}{2}$$

(This was done after class)

$$\# \text{snakes} = \underbrace{\underbrace{(n-1)}_{\substack{\#\text{ snakes} \\ \text{with head} \\ \text{on } n}} + \underbrace{(n-2)}_{\substack{\#\text{ snakes} \\ \text{with head} \\ \text{on } n-1}} + \cdots + \underbrace{1}_{\substack{\#\text{ snakes} \\ \text{with head} \\ \text{on } 2}}}_{(n-1) \text{ sets of snakes}}$$

Why did we add?

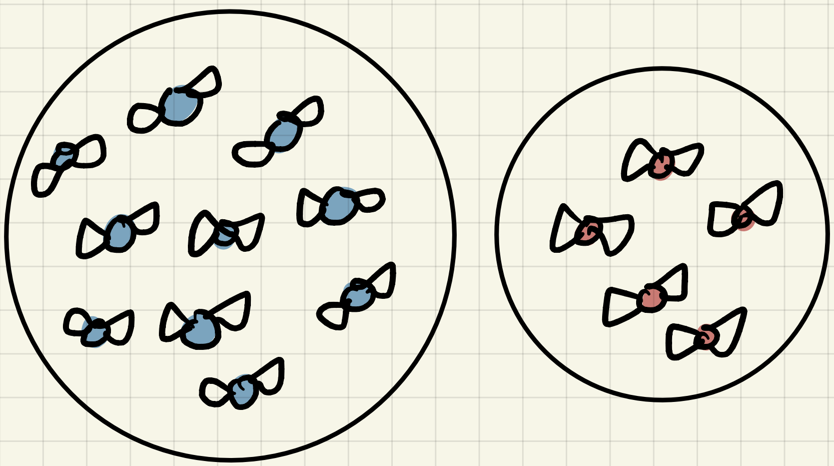
Read ch. 0 Section 11 on sets

Informally, a set S is a collection of things called elements. If S is finite, $|S|$ denotes its "cardinality", which is the # elements in it.

Addition Rule: Given k sets, $S_1, S_2, \dots, S_k$ that are pairwise <u>disjoint</u>, the total # of elements in their union is

$$|S_1| + |S_2| + \cdots + |S_k| = \sum_{i=1}^{k} |S_i|$$

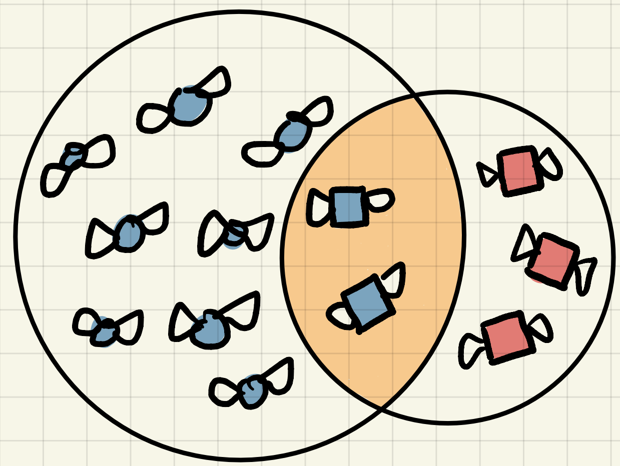
I have 10 blue and 5 red candies



Disjoint

total $= 10 + 5 = 15$

I have 10 blue and 5 square candies



Not Disjoint

total $\neq 10 + 5$

Another way to count snakes

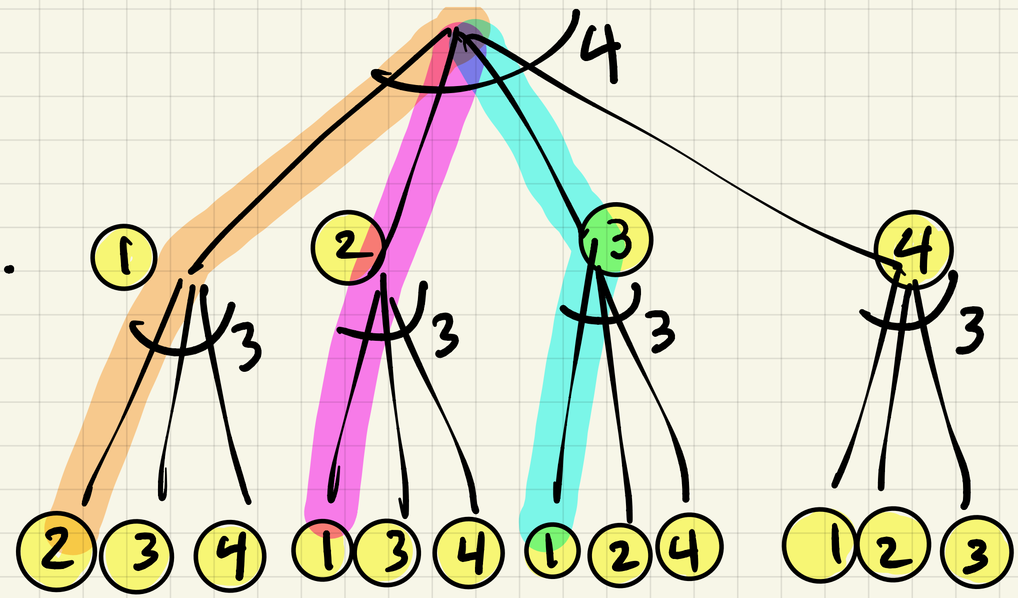
"To count something think about a task to generate one possible outcome"

Example: $n=4$

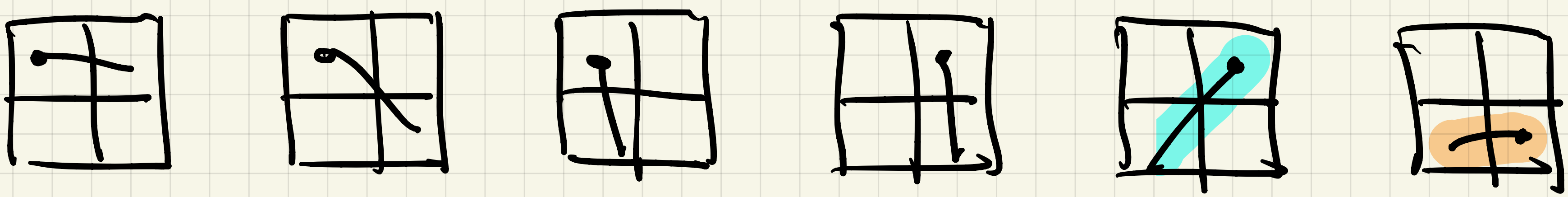
4	3
1	2

1. choose a square .....

2. choose another square ...



$4 \times 3 = 12$



Product rule: If a task consists of k phases and each phase i can be done in α_i ways regardless of other phases, then the total task can be carried out in

$$\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_k = \prod_{i=1}^{k} \alpha_i \quad \text{ways}$$

Example: generating a snake consists of two phases.

- Phase 1 can be done in $\alpha_1 = 4$ ways
- Phase 2 can be done in $\alpha_2 = 3$ ways (regardless of phase 1)

But why don't we have $4 \times 3 = 12$ snakes?

⇒ Because the procedure **overcounts**!

For example $(1,2)$ and $(2,1)$ represent the same snake. If fact, every snake is counted exactly twice.

Adjusting for overcount: $\dfrac{4 \times 3}{2} = 6$

It's good to overcount if you know by how much

Counting snakes for general n using product rule:

1. Choose a square . . . . . . n ways
2. choose another square . . . . . $n-1$ ways

$$n(n-1)$$

Each snake is counted exactly twice

(a, b) and (b, a) are "physically" different but equivalent in our setting.

$$\text{So } \#\,\text{snakes} = \frac{n(n-1)}{2}$$

[with product rule, adjust for overcount if any]