Putting the Product rule to work

Recall :

\# permutations on n objects .... $n!$

why?

Think of task for generating one permutation

		# ways
1.	choose an object for 1^{st} position	n
2.	choose another object for 2^{nd} position	$n-1$
⋮		
k.	choose another object for k^{th} postion	$n-k+1$
⋮		
n.	choose another object for n^{th} position	1

$$n(n-1)(n-2)\cdots 1 = n!$$

We can't permute choices in phases and get same outcome

$\Rightarrow$ No overcounting

In how many ways can we seat
n people on n chairs?

Find a task to generate a seating

Abstraction

This is a permutation

$n!$

Recall :

\# pairs on n objects .... $\binom{n}{2} = \frac{n(n-1)}{2}$

Think of task for generating one pair

	# ways
1. choose an object	n
2. choose another object	$n-1$
	$n(n-1)$

Is order relevant ?

Left/Right sock : Yes , No overcount $\Rightarrow n(n-1)$

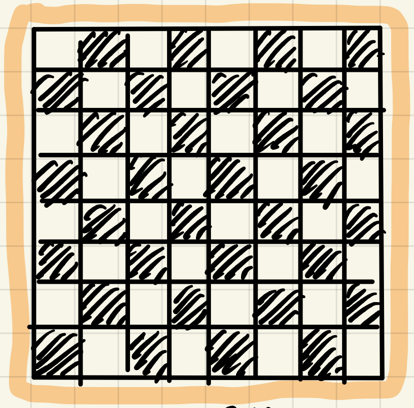
Snake : No, overcount by 2 $\Rightarrow \frac{n(n-1)}{2}$

What did we learn ? So far, when we talked about pairs we meant <u>unordered</u> .

\# ordered pairs $= n(n-1)$.

\# unordered pairs $= \binom{n}{2} = \frac{n(n-1)}{2}$

Snakes & Ladders on a chessboard



$n = 64$

In how many ways can we place one snake if head & tail must be on different colors? (Assume n is even)

	#ways
1. choose a square	n
2. choose diff. color square	$n/2$
	$n \times \frac{n}{2}$

Can we permute the choices and get same outcome? Yes

overcounting by 2,

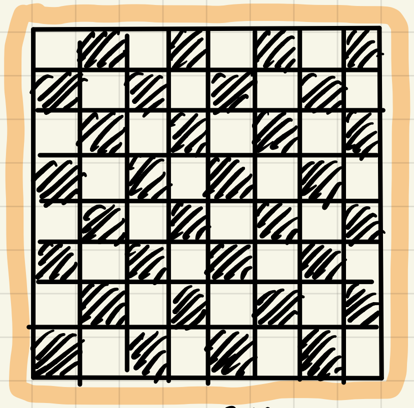
so answer $= \frac{n}{2} \times \frac{n}{2} = \frac{n^2}{4}$

	#ways
1. choose a black square	$\frac{n}{2}$
2. choose a white square	$\frac{n}{2}$
	$\frac{n}{2} \times \frac{n}{2}$

Can we permute the choices and get same outcome? No

answer is as before $\frac{n^2}{4}$

Snakes & Ladders on a chessboard



$n = 64$

In how many ways can we place one snake if head & tail must be on same color? (Assume n is even)

	#ways
1. choose a square	n
2. choose same color square	$\frac{n}{2}-1$
	$n\left(\frac{n}{2}-1\right)$

Can we permute the choices and get same outcome? Yes: $\frac{n}{2}\left(\frac{n}{2}-1\right)$

	#ways
1. choose a black square	$\frac{n}{2}$
2. choose diff. black square	$\left(\frac{n}{2}-1\right)$
	$\frac{n}{2}\left(\frac{n}{2}-1\right)$

Can we permute the choices and get same outcome? Yes: $\frac{n}{4}\left(\frac{n}{2}-1\right)$

why $\neq$?

(see below)

Each of the following two task generates only parts of total possible outcomes

1. Choose a black square ...... $\frac{n}{2}$

2. choose diff. black square ... $\left(\frac{n}{2}-1\right)$

$$\frac{n}{2}\left(\frac{n}{2}-1\right)$$

1. Choose a white square ...... $\frac{n}{2}$

2. choose diff. white square .... $\left(\frac{n}{2}-1\right)$

$$\frac{n}{2}\left(\frac{n}{2}-1\right)$$

black

white

Disjoint

Adjust for overcounting

$$\frac{n}{4}\left(\frac{n}{2}-1\right) \qquad \frac{n}{4}\left(\frac{n}{2}-1\right)$$

$+$

Addition rule: $\frac{n}{2}\left(\frac{n}{2}-1\right)$ ✓

In how many ways can we place two snakes?

As usual, think of a task that generates two snakes by making choices

		# ways
snake 1	1. choose a square	n
	2. choose another square	$(n-1)$
snake 2	3. choose another square	$(n-2)$
	4. choose another square	$(n-3)$
		$n(n-1)(n-2)(n-3)$

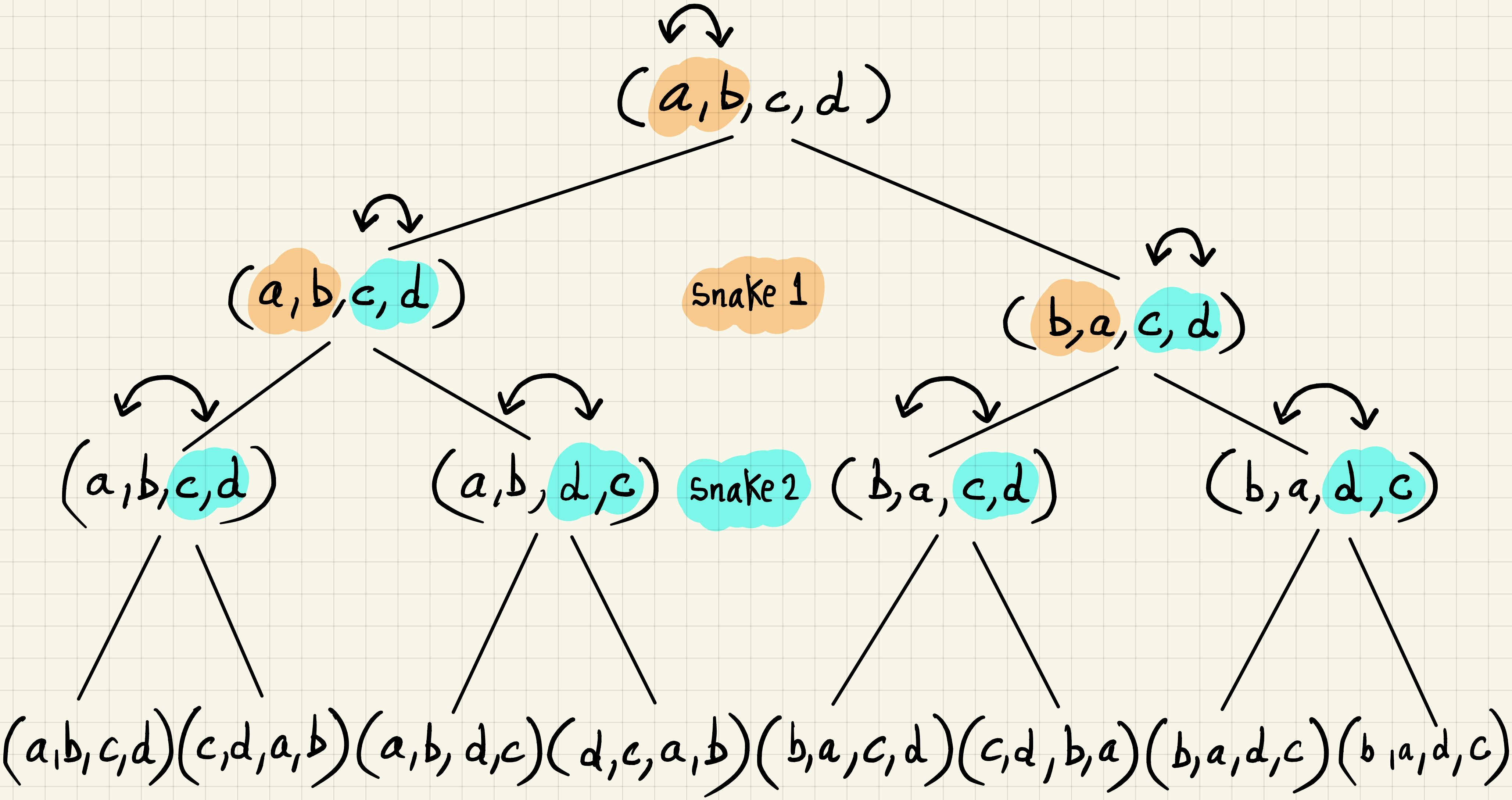
2

2

2

Is there overcounting?

YES!

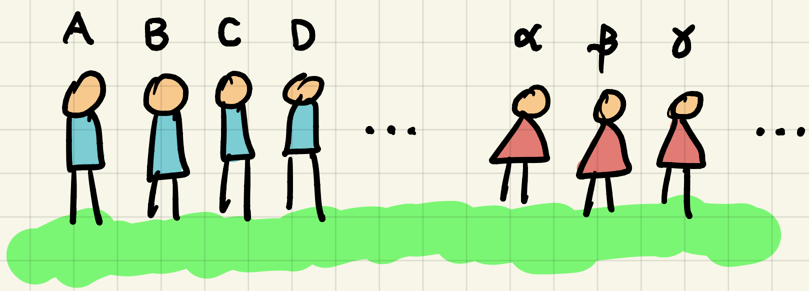


Overcounting by $2 \times 2 \times 2 = 8$

Answer: $$\frac{n(n-1)(n-2)(n-3)}{8}$$

Boys and Girls

Given m boys and n girls, in how many ways can we make a couple?



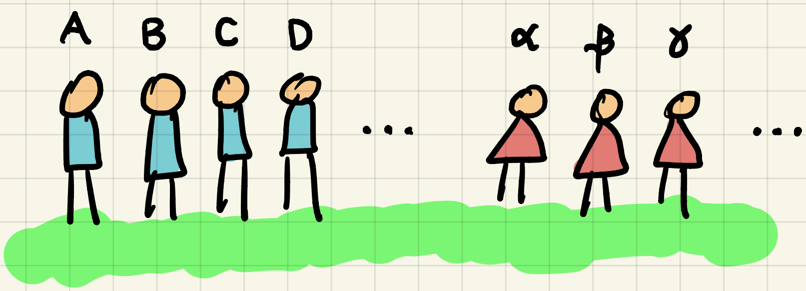
	# ways
1. choose a person	$m + n$
2. choose a diff. gender person	?

The number of ways for 2^{nd} phase is not independent of choices in 1^{st} phase!

(product rule does not work here)

Boys and Girls

Given m boys and n girls, in how many ways can we make a couple?



	# ways
1. choose a boy	m
2. choose a girl	n
	$m \times n$

Is there overcount? No, phases cannot be permuted; for instance, phase 1 cannot generate girl

Unordered pairs

$$\left.\begin{array}{ll} \text{\# pairs of boys :} & \binom{m}{2} \\ \text{\# pairs of girls :} & \binom{n}{2} \\ \text{\# couples :} & mn \end{array}\right\} \text{Disjoint}$$

$$\text{\# pairs :} \quad \binom{m+n}{2}$$

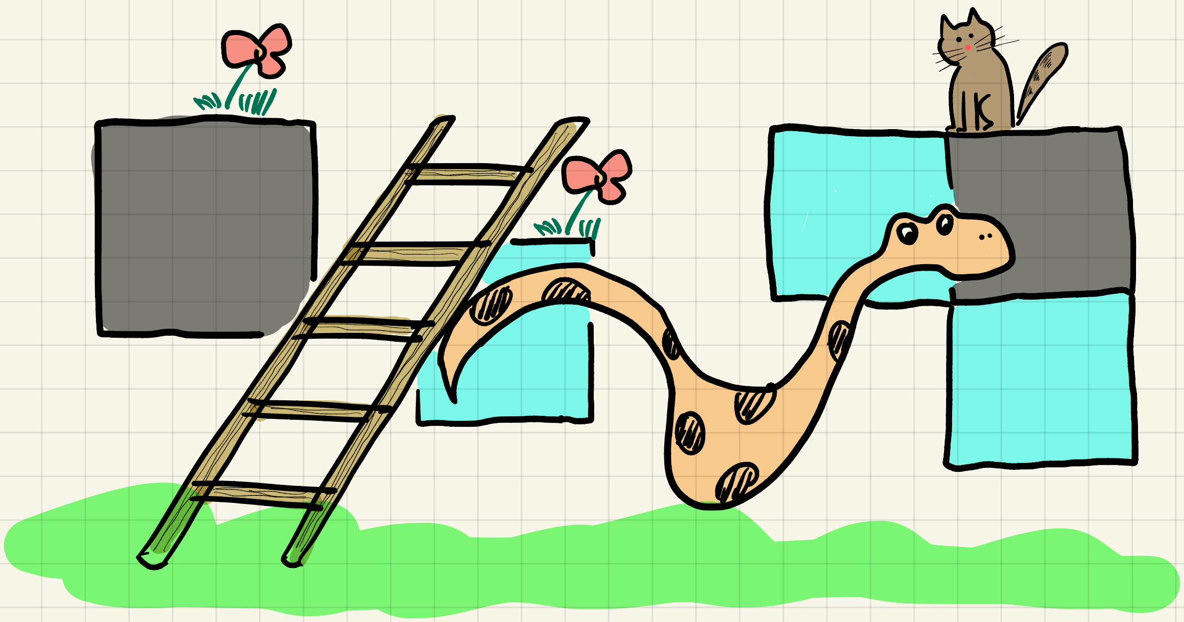
What does the addition rule tell us?

$$\binom{m}{2} + \binom{n}{2} + mn = \binom{m+n}{2}$$

Verify: $\dfrac{m(m-1)}{2} + \dfrac{n(n-1)}{2} + mn = \dfrac{(m+n)(m+n-1)}{2}$

Exercise:

In how many ways can we place one ladder and one snake on a chessboard if the head and tail of the snake must be on different colors?



Hint: Why should we place the snake first?